Chapter II

Fundamentals Knowledge

2.1 Historical Development of Comminution

Comminution is one of the oldest mechanical unit operations of powder processing for the purpose of size reduction of a solid mass used to generate a large quantity of particulate materials. Its importance is now even more growing. It not only makes solids handling easier, but also improve physiochemical properties substantially, due to large increment of surface area. This is utilized to raise the efficiency of subsequent processes such as to mix different types of materials, to recycle useful components after separation.

Despite its long history, fundamental analysis and optimum operation have been poorly developed because of the complex mechanism. Most energy input is not transmitted to ground solids and emitted as loss generally in form of heat, due to the ineffective motion of grinding media and the internal friction of particle assemblage. Thus a tremendous amount of energy is consumed at a very low efficiency.

2.2 Classification in Comminution Process

Generally, there are so many words involved with comminution. Regarding size range of ground solid, they are commonly classified as follows,

Crushing	order of cm-size	→	mm-size
Grinding	order of 10^2 mm-size	→	10^2 μm-size
Fine grinding	order of 10^1 mm-size	→	10^0 μm-size
Ultra fine grinding	order of 10^2 μm-size	→	submicron size

However, this is not established definition. For instance Schonert's definition (1990) is completely different as follows.

Fine grinding → below 100 ~ 300 µm
Very fine grinding → below 10 ~ 30 µm
Micro fine grinding → below 1 ~ 3 µm

It is not important to conclude the definitions above mentioned, but just to give an idea of very wide range which comminution process should cover.

2.3 Process of Comminution

The comminution of solids involves operations which aim not only reducing the dimensions of solid materials from coarser sizes but also producing materials with morphological shape. Crushers and grinders are types of comminuting equipment. An ideal crusher or grinder would

1. Have a large capacity
2. Require a small power input per unit of product, and
3. Yield a product of the single size or the narrowest size distribution desired, or the specific shape.

Most solids are crystalline materials in which the atoms are regularly arranged in three-dimensional arrays. The configuration of atoms is determined by size and types of physical and chemical bonds holding them together. In the crystalline lattice of solids, these inter atomic bonds are effective only over small distances, and can be broken if extended by a tensile stress. Such stresses may be generated by tensile or compressive loading (Figure 2.1)

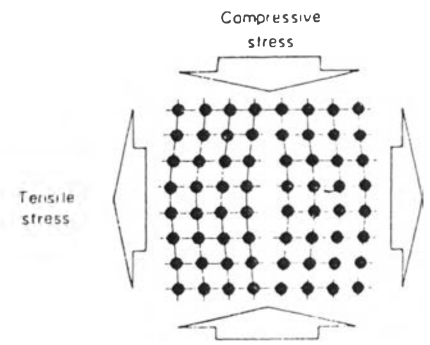


Figure 2.1 Strain of a crystal lattice resulted from tensile or compressive stresses.

When the particles are loaded by tensile or compressive stress, the internal stresses will occur inside them. If the internal stresses are evenly distributed. cracking in the solid body will take place. Cracking of particles can be discussed by relationship of Stress-Strain curve explained by Hooke 's Law. (Figure 2.2)

In the fact that the internal stresses are not evenly distributed, even when the particles are uniformly loaded. The distribution of stress depends upon the mechanical properties of the individual parts inside the particles, but more importantly, upon the presence of cracks or flaws in the matrix, which act as sites for stress concentration (Figure 2.3)

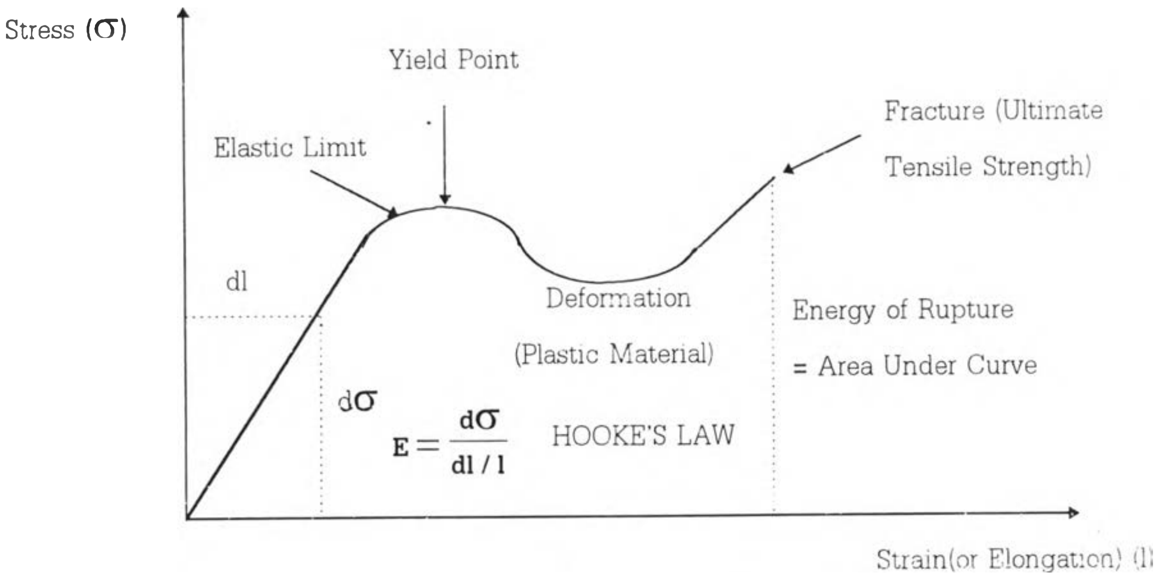


Figure 2.2 The relationship between Stress-Strain of Materials

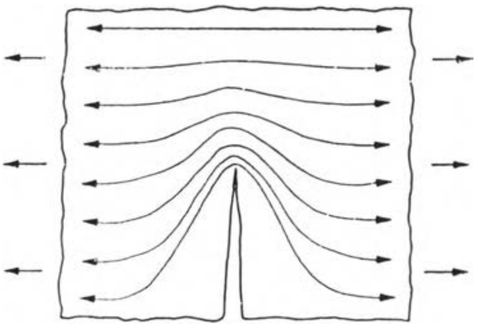


Figure 2.3 Stress concentration at a crack tip

In has been shown that the increase in stress at such a site is proportional to the square root of the crack length perpendicular to the stress direction. Therefore there is a critical value for the crack length at any particular level of stress at which the increased stress level at the crack tip is sufficient to break the atomic bond at that point. Such rupture of the bond will increase the crack length, thus increasing the stress concentration and causing a rapid propagation of the crack through the matrix, thus causing fracture.

2.4 The Law of Comminution Energy

The equation to describe the behavior of crack of solids when it is forced by energy input can be expressed in term of ratio between feed particle size before comminution process and product particle size after comminution process. This ratio is called reduction ratio which follow by equation 2.1

$$Reduction \quad Ratio \quad = \quad \frac{Size \quad of \quad Feed}{Size \quad of \quad Product} \qquad (2.1)$$

There are many researchers proposed various theories, but none of which is entirely satisfactory. Because it can not predict result which occur from comminution accurately. The greatest problem is most of the energy input in comminution machine is absorbed by the machine itself, and only a small fraction of the total energy is available for breaking the material. It is to be expected that there is a relationship between the energy required to break the material and the new surface produced in the process, but this relationship can only be made manifest if the energy consumed in creating new surface can be separately measured.

1. Rittinger's theory (1867) This theory assumes that the energy consumed in the size reduction is proportional to the area of new surface produced. The surface area of a known weight of particles with uniform size is inversely proportional to the diameter, hence Rittinger' s theory equates to

$$E_R = K_R\left(\frac{1}{D_P} - \frac{1}{D_F}\right) \tag{2.2}$$

$$= K_R^{/}\left(S_P - S_F\right) \tag{2.3}$$

where	D_P and D_F	are product particle size and feed particle size respectively,
	S_P and S_F	is specific surface of product and feed respectively, and
	K_R and $K_R^{/}$	are constants

2. Kick's theory (1885) Kick has explained that the workdone required is proportional to the reduction in volume of the particle concerned. Where D_F is the diameter of the feed particles and D_P is the diameter of product particles , the reduction ratio R is D_F / D_P . So the equation of Kick , which calculate the energy required for comminution is expressed as follow,

$$E_K = K_K \log \frac{D_F}{D_P} \tag{2.4}$$

$$= K_K \log R \tag{2.5}$$

$$= K_K^{/} \log \frac{S_P}{S_F} \tag{2.6}$$

where	K_K and $K_K^{/}$	are constants
	D_F and D_P	are product particle size and feed particle size respectively, and
	S_P and S_F	are the specific surface of product and feed respectively

3. Bond's theory (1952) Bond developed an equation based on the theory that the necessary work input is proportional to the new crack tip length produced in particle breakage. In particles of similar shape , the specific surface area of material is inversely proportional to the diameter. The crack length in unit volume is considered to be proportional to one side of that area and therefore inversely proportional to the square root of the diameter

This relationship can be showed in this equation.

$$E_B \quad \alpha \quad \frac{1}{\sqrt{D_P}} - \frac{1}{\sqrt{D_F}} \qquad (2.7)$$

$$\text{or} \quad E_B = K_B \left(\frac{1}{\sqrt{D_P}} - \frac{1}{\sqrt{D_F}} \right) \qquad (2.8)$$

$$\text{or} \quad E_B = K_B^{\prime} \left(\sqrt{S_P} - \sqrt{S_F} \right) \qquad (2.9)$$

where E_B is the total energy of comminution in kilowatt hours per short ton.

K_B and $K_B^{\prime}$ are constant.

The diameter in microns which 80% of the product passes is designated as P_{80}, and the size which 80% of the feed passes is designated as F_{80} . The Bond's equation can be also expressed as

$$E_B = K_B \left(\frac{1}{\sqrt{P_{80}}} - \frac{1}{\sqrt{F_{80}}} \right) \qquad (2.10)$$

Bond defined term of work index W_i, where as the work input is in kilowatt hours per short ton. The work index is the comminution parameter which expresses the resistance of the material to comminuting; numerically it is the kilowatt hours per short ton required to reduce the material from theoretically infinite feed size(α) to 80% passing 100 µm. Hence, from equation (2.10),

$$W_i = K_B \left(\frac{1}{\sqrt{100}} - \frac{1}{\sqrt{\alpha}} \right) \tag{2.11}$$

$$W_i = K_B \frac{1}{\sqrt{100}}$$

or $$K_B = 10 \; W_i \tag{2.12}$$

Therefore, to grind particles in size from F_{80} to P_{80}, the power requirement for size reduction is given by equation 2.10 and 2.12 as

$$E_B = W_i \left(\frac{10}{\sqrt{P_{80}}} - \frac{10}{\sqrt{F_{80}}} \right) \tag{2.14}$$

2.5 Shape Characterization of Particles

2.5.1 Introduction

Characterization of the materials which are in the powdered state is an essential to ensure its quality. The characteristics of powder are influenced by the material, method of powder production and the process variables in a particular method. Consequently such characteristics influence its behavior during handling and processing such as the surface area which affect the chemical reaction. These characteristics have been divided under two broad categories as shown in Figure 2.4

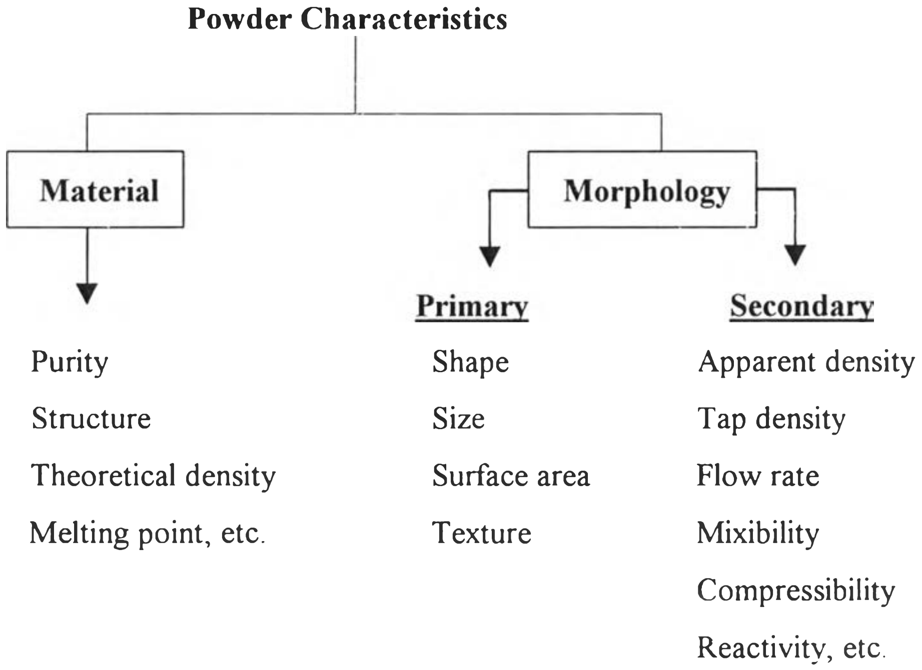


Figure 2.4 A broad classification of powder characteristics (Paramanand, 1996)

One of the characteristics of the particles which can be seen from the above figure is the particle shape. This characteristics is increasingly recognized as an major variable in predicting the behavior of both individual particles and particles system. Nevertheless, characterizing particle's shape has been a problem due to a lack of definition of shape and methods for defining particle shape.

2.5.2 Characterization of particle shape by Fractal Dimension

Fractal, which comes from the Latin adjective "fractus", has the same root as "fraction" and "fragment" and means "irregular or fragmented", defined by Mandelbrot (1977). Mathematically a fractal is a term used to describe an object, which has a dimension of non-integer ,so- called fractal dimension. Such a parameter describes the deviation that a line, surface, or volume deviate from a topological ideal. For example, an ideal topological line has a dimension equal to 1, and an ideal plane has a topological dimension equal to 2. A fractal line, though, has a fractal dimension greater than 1, but less than 2, and a fractal surface has a fractal dimension greater than 2 but less than 3.

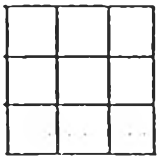
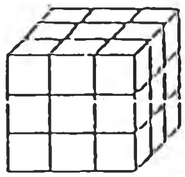
The fractal dimension is a non integer (real number) that represents the dimensionality of that fractal. Even a fractal normally considered as one-dimensional, for example, a line segment, also posses a self-similar scaling property. The line segment can be divided into N identical parts, each of which is scaled down by the ratio r = 1/N from the whole. Similarly, a two-dimensional object, such as a square area on a plane, can be divided in N self-similar parts, each of which is scaled down by a factor $r = 1/N^{(1/2)}$. A three-dimensional object, such as a solid cube, can be divided in N little cubes, each of which is scaled down by ratio $r = 1/N^{(1/3)}$

With self-similarity the generalization to fractal dimension is straight forward. A D-dimensional self-similar object can be divided in N smaller copies of itself, each of which is scaled down by a factor r, where $r = 1/N^{(1/D)}$. Conversely, a self-similar object of N parts scaled by a ratio r from the whole, as given in table 2.1, has a dimension, which can be expressed as follows :

$$D = \frac{d\log N}{d\log(1/r)} = -\frac{d\log(N)}{d\log(r)}$$

In other words, the fractal is defined as an extremely irregular line (or surface) made up of an infinite number of similarly irregular sections (or parts). The fractal has fractal dimension between one and two for line segments (or between two and three for surface). It could be shown that an object with extremely irregular shape might be treated as a fractal.

Table 2.1 Scaling law for fractal dimension (Sirikalaya, 1994).

Object	Number of pieces (N)	Scaling (r)	Law
	3	1/3	$3 = 3^1$
	9	$1/9^{(1/2)} = 1/3$	$9 = 3^2$
	27	$1/27^{(1/3)} = 1/3$	$27 = 3^3$

Let N(r) be the counted number of subsections (squares of side r) containing at least one small part of a fractal when the representative length of each subsection was r , r = 1/n (n = number of division to similarity) , the so-called similarity ratio.

For the fractal, the following equation holds :

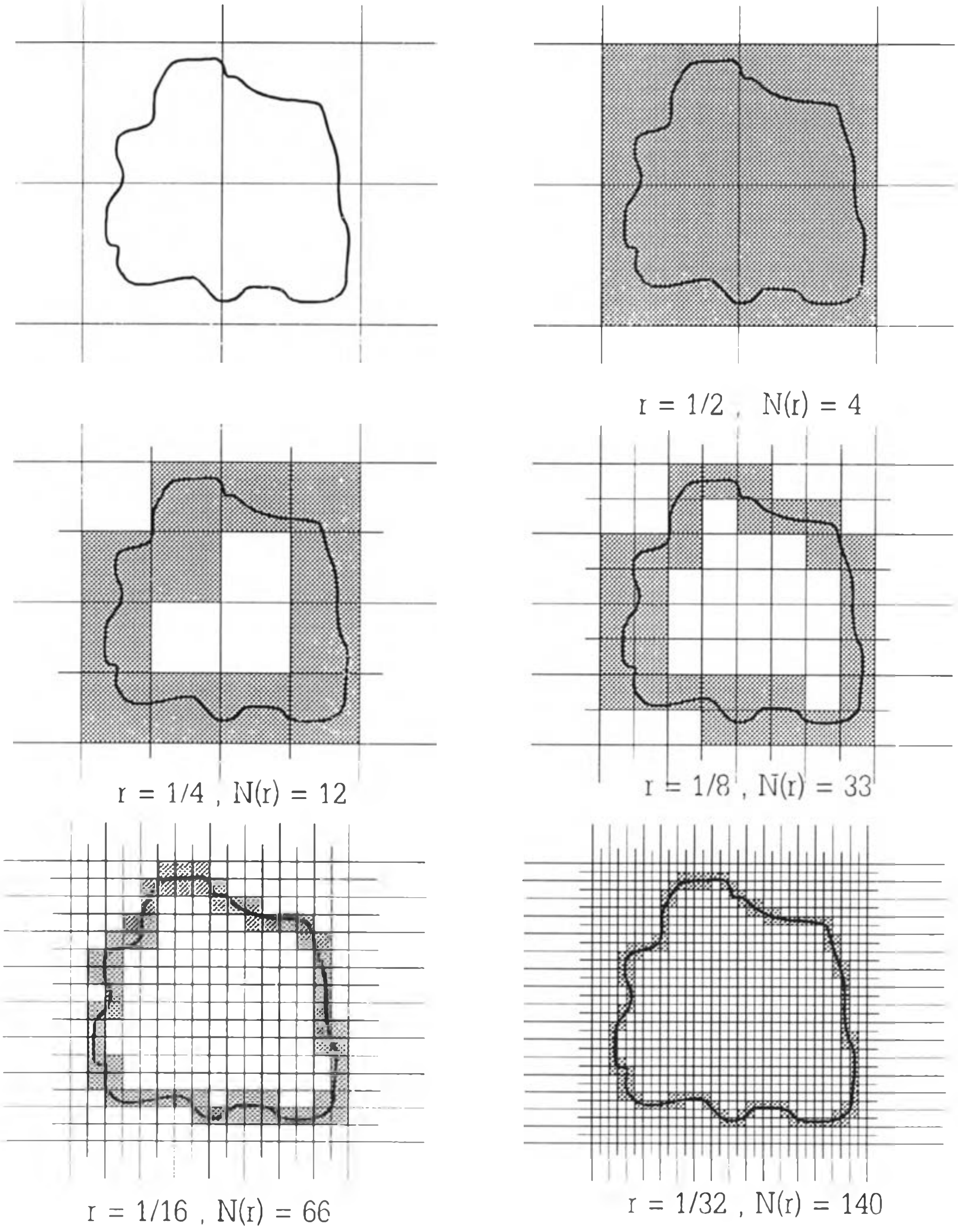
$$N(r) = (Similarlity\ \ Ratio)^{-D} = (r)^{-D}$$

Thus the fractal dimension, D, is defined as:

$$D = \frac{-d \log N(r)}{d \log r}$$

2.5.3 Fractal Analysis by Count Method

In the fractal analysis of particle shape in this study used the count method. An illustration counting the number of subsections, N (r), that have the edge of the particle versus the similarity ratio r divided is shown in Figure 2.5



r = 1/2 , N(r) = 4

r = 1/4 , N(r) = 12

r = 1/8 , N(r) = 33

r = 1/16 , N(r) = 66

r = 1/32 , N(r) = 140

Figure 2.5 Determination of fractal dimension of particle shape

The observed relationship between N(r) and r was plotted on the log-log scale, and the fractal dimension can be calculated from the slope of the convergent straight lines as shown in Figure 2.6

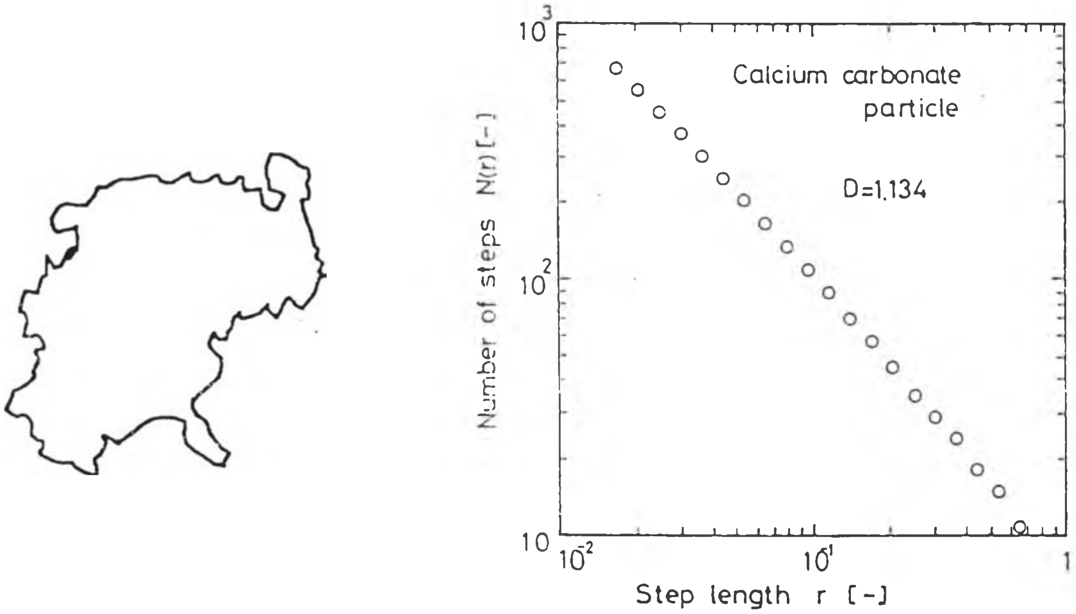


Figure 2.6 Particle shape profile and a graph for determining fractal dimension of a calcium carbonate particle (Suzuki,1990)

2.6 Evaluating Flow Properties of Solids

In the process which concerns particle, the chemical engineer often faces the problem of handling these particles. Here described are a number of standard procedures that will permit the evaluation of flow characteristics of any materials. It will also provide information on other properties of materials that can affect the handling problem.

2.6 Evaluating Flow Properties of Solids (Ralph L. Carr, 1965)

2.6.1 Evaluating Flowability

The evaluation of the flow characteristics of a particle will involve the use of four properties :

1. Angle of Repose
2. Compressibility
3. Angle of Spatula
4. Cohesion or Coefficient of uniformity

Angle of Repose

The angle of repose defined by Terzaghi (1948) is the angle between the horizontal and the slope of a heap of particle dropped from some elevation. For our purposes, it can be defined as the constant angle to the horizontal assumed by a conelike pile of particle. This pile is carefully built up by dropping particle from a point above the horizontal plane until a constant angle is measured. Figure 2.7 shows an apparatus which is used to measure the angle of repose.

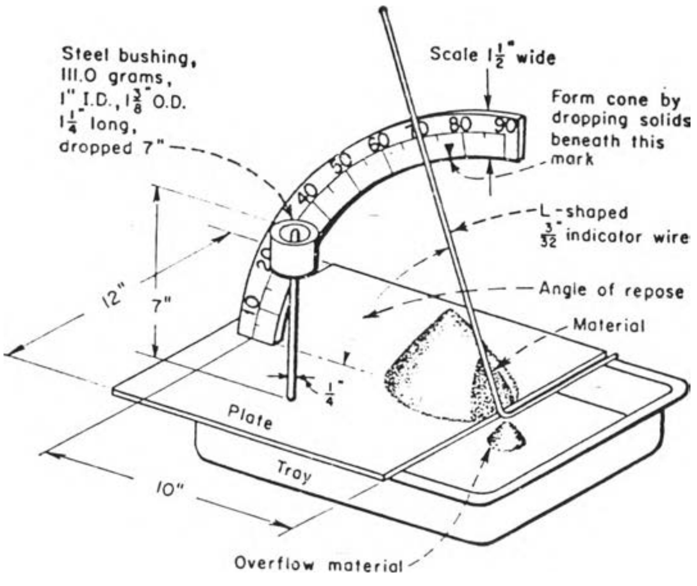


Figure 2.7 The angle of repose measurement.

The lower the angle of repose of a particle, the more flowable it will be, and the more floodable it will be. Because it occurs from the friction force between particles.

Compressibility

The percentage compressibility determined by the measurement of aerated and packed bulk density are defined as follows,

$$\frac{100\ \ (P-A)}{P} = \%\ \ Compressibility$$

where P = Packed bulk density (g/cu. cm)

A = Aerated bulk density(g/cu. cm)

Angle of Spatula

A spatula, which is a 5 x (7/8) in. blade is inserted parallel to the bottom of the container and then lifted straight up. A free-flowing material will form a pole on the blade. A non-free-flowing material will form a number of irregular angles of rupture on the blade. In either case, the angles to the horizontal are measured and an average value is to be taken. Then the spatula is tapped gently, producing a lower angle or angles of spatula, which are measured and again averaged. The average of two measurements is used in the flow computation and is termed as the angle of spatula.

Except for very free-flowing materials, the angle of spatula is always higher than the angle of repose for a given powder. The angle of spatula gives a broader spectrum of flowability information than the angle of repose. The higher the angle of spatula of a powder, the less its flowability will be. For a powder to be considered free flowing, its angle of spatula should be under 40 degree. Powder with a very non-uniformity in both size and shape will need a broader width of blade on which to form an angle of spatula.

Cohesion and Uniformity

Cohesion and the uniformity coefficient are used in flow evaluation. Cohesion is used with powders (very fine particles) or with materials on which an effective cohesive force can be measured. The uniformity coefficient is used with granular and powdered.

Cohesion. The procedure for finding the apparent surface cohesion involves determining the retention of material on a nest of 60,100,and 200-mesh screens, plus a bottom pan. This test is a direct determination of the amount of energy necessary to pull apart aggregates of cohesive particles in a specified time.

Uniformity Coefficient. This is the numerical value arrived at by dividing the width of sieve opening that will pass 60% of the sample by the width of sieve opening that will just pass 10% of the sample. It is determined from the screen analysis of the material.

2.6.2 Kinds of Bulk Density

The bulk density of a material is the weight per unit volume of many particles of the material. The unit is customarily pounds per cubic foot. Specific gravity, on the other hand, is a relative number relating the mass and the volume of distilled water at a certain temperature. Except for its effect on compressibility, bulk density is not useful in evaluating the flow of a substance. There is no direct linear relationship between the potential flowability of a powder and its bulk density. Other properties of the substance counteract whatever potential effect its bulk density could have on flowability.

The bulk density of a material is not a single definite number like specific gravity; rather it is any one of several values. Four different values are computed for the bulk density of a material and one more is indicated.

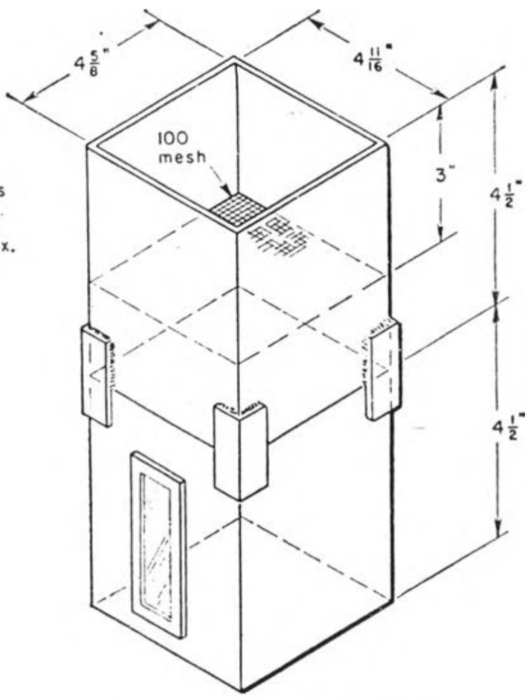
The methods for obtaining the aerated (loose) and packed bulk density are described below. Other terms include average bulk density, which is simply the average of the aerated and packed bulk densities, and the working bulk density, which is

$$W = (P-A)C+A$$

where W, P and A are respectively working, packed, and aerated bulk densities, and C is compressibility expressed as a fraction. Working bulk density is the figure most often used in computations.

Fluid Bulk Density. This is less than the aerated bulk density of a material. The faster a fluidlike material is moved through air, the more fluidlike it can become and the greater the volume it will occupy. Thus its weight per unit volume will be less in every case than the aerated bulk density of a material, although the exact difference is hard to pin down because of the variable factors involved.

Measuring Bulk Density. Select a small metal cylinder or square container (or even a wide-mouthed jar). This will be used, with an appropriate factor, to relate directly the net weight of the material in this small container to its actual bulk density in pounds per cubic foot. Fig.2.8 shows the design of container used. The less uniform a material is in its range of particle size, the larger a container should be used.



Dimensions are inside.

Vol. approx. 100 cu.in.

Net wt.(lb.) in lower box X 17.4 = lb./cu.ft.

Figure 2.8 Bulk -density box for testing both aerated and packed bulk densities.

Aerated (Loose) Bulk Density . Carefully screen the material passing the coarse screen (about 10 mesh) into the bulk-density container below. Fill material to overflowing . Level off the overflow and weight the container. The net weight times an the aerated or loose bulk density in pounds per cubic foot.

Packed Bulk Density (Tap Bulk Density). Fill the material into the bottom container and put more material into the top container. Put the two containers on the tapping holder, The standard number of tapping counts is 180 counts. The material in the lower container packs down, leaving a void that is filled with material tapped. After standard tapping, remove the top container, level off the excess in the bulk-density container and weight it. The vibrated net weight gives the packed bulk density.

2.6.3 Calculating Flowability

Table 2.2 shows how these various criteria can be used in calculating a flowability point score for any type of dry material. Through experience gained in testing over 2,800 dry-material samples, we are able to divide the point scores of materials into seven categories, ranging from excellent to very, very poor from a flowability standpoint. These categories are explained in the first, second and third column of Table 2.2. In each of the "property" columns, the values of the property are listed, along with an equivalent point score for each. Thus the ideal value of each property has a point score of 25. As the value of the property increases, the point score decreases, finally reaching zero for the poorest material.

It will be noted that the best material in the bottom category of Table 2.2 would receive a point score of only 18. Any material with a point score under 20 requires special equipment and perhaps special engineering in order to feed it. Standard feeders will not simply be applicable.

After measuring the flow properties of a material, one goes to Table 2.2 and allots points according to each flow property value of the material.

Table 2.2 Evaluation of flowability index of powder

Degree of Flowability	Flowability Index	Necessity of Bridge-breaking measures	Angle of Repose		Compressibility		Angle of Spatula		Uniformity *		Cohesion **	
			Degree	Index	%	Index	Degree	Index	No.	Index	%	Index
Very Good	90-100	Not required	<=25	25	<=5	25	<=25	25	1	25		
			26-29	24	6-9	23	26-30	24	2-4	23		
			30	22.5	10	22.5	31	22.5	5	22.5		
Fairly Good	80-89	Not required	31	22	11	22	32	22	6	22		
			32-34	21	12-14	21	33-37	21	7	21		
			35	20	15	20	38	20	8	20		
Good	70-79	Sometimes Vibrator is required	36	19.5	16	19.5	39	19.5	9	19		
			37-39	18	17-19	18	40-44	18	10-11	18		
			40	17.5	20	17.5	45	17.5	12	17.5		
Normal	60-69	Bridging will take place at the marginal point	41	17	21	17	46	17	13	17		
			42-44	16	22-24	16	47-59	16	14-16	16		
			45	15	25	15	60	15	17	15	<=6	15
Not Good	40-59	Required	46	14.5	26	14.5	61	14.5	18	14.5	6-9	14.5
			47-54	12	27-30	12	62-74	12	19-21	12	10-29	12
			55	10	31	10	75	10	22	10	30	10
Bad	20-39	Powerful measure should be provided	56	9.5	32	9.5	76	9.5	23	9.5	31	9.5
			57-64	7	33-36	7	77-89	7	24-26	7	32-54	7
			65	5	37	5	90	5	27	5	55	5
Very Bad	0-19	Special apparatus and techniques are required	66	4.5	38	4.5	91	4.5	28	4.5	56	4.5
			67-89	2	39-45	2	92-99	2	29-35	2	57-79	2
			90	0	>=45	0	>=99	0	>=35	0	>=79	0

* Use these figure for granules or granular powder with which the uniformity can be measured.

** Apply these figures for fine and cohesive powders with which the cohesion can be measured.

2.6.4 Evaluating Floodability

Floodability of a material is its tendency to liquid like flow due to natural fluidization of a mass of particles by air. The evaluation of the potential floodability of a material involves the use of the following four properties of dry materials :

1. Flowability
2. Angle of fall
3. Angle of difference
4. Dispersibility

Flowability

This is the same property evaluated above. The more flowable a potentially floodable material is, the greater its floodability can be. A material with a flowability point score of, say, 58 would tend to form a relatively weaker arch in a hopper and thus would tend to flood more readily from this weaker arch than a material of lower point score.

Angle of Fall

After carefully building up the angle of repose of a substance on a metal plate equipped with a protractor for angle-of-repose measurements, a small steel bushing of 111.0 grams weight is dropped on to the plate three times from a height of 7 in. (See Figure 2.7). A new angle of repose results from the jarring. This angle is called the "angle of fall". The angle of fall is a direct measure of the relative flow and floodability of a material.

Angle of Difference

This is the value arrived at by noting the difference between the angle of repose and the angle of fall. Some floodable materials may not have a very low angle of fall but may still have a high angle of difference. This measurement gives still another picture of the potential floodability of a material apart from its angles of repose and fall.

The greater the angle of difference of a floodable material, the greater its potential for flooding or fluidizing.

Dispersibility

Dispersibility, dustiness and floodability are all interrelated. The more dispersible a powder is in air, the more dusty and the more floodable it can be.

Dispersibility is a direct measure of the ability of a material to flood or be fluidized. To determine dispersibility, an appratus consisting of a 4-in.-I.D. plastic cylinder with length of 13 in. supported vertically with a ring-stand above a 4-in.-dia. Watch glass and prepared 10 grams of material. Then the material is dropped through the cylinder from a height of 24 inch above the watch glass. Material remaining on the watch glass is weighted. Some material has been dispersed during the fall. After that calculate the % dispersibility from this equation

$$\frac{[10 - (\text{weight in grams remaining})] \times 100}{10}$$

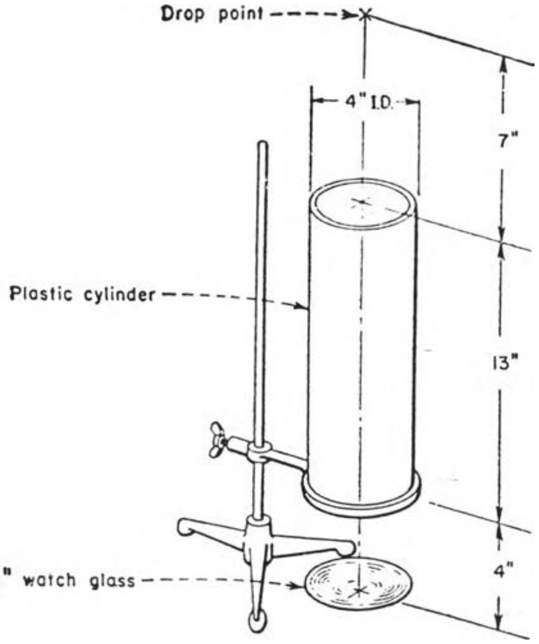


Figure 2.9 Dispersibility test uses a plastic cylinder through which the sample is dropped onto a watch glass below.

2.6.5 Calculating Floodability

The values of floodable properties are listed in the second column of Table 2.3. A hypothetical material could total 100 points. In Table 2.3, point scores are distributed, and floodability scores are divided into categories in a manner similar to Table 2.2.

This floodability evaluation procedure gives an all around picture of the potential floodability of materials. A very floodable material is seen to have passable to fair-but unstable-flowability, a low angle of fall, a large angle of difference, and high dispersibility.

Table 2.3 Evaluation of floodability index of powder

Degree of Floodability	Floodability Index	Measures for Flushing Prevention	Flowability		Angle of Fall		Angle of Difference		Dispersibility	
			Index from Table 2	Index	Degree	Index	Degree	Index	%	Index
Very high	80-100	Rotary seal must be used	>=60	25	<=10	25	>=30	25	>=50	25
			56-59	24	Nov-19	24	28-29	24	44-49	24
			55	22.5	20	22.5	27	22.5	43	22.5
			54	22	21	22	26	22	42	22
			50-53	21	22-24	21	25	21	36-41	21
			49	20	25	20	24	20	35	20
Fairly High	60-79	Rotary seal is required	48	19.5	26	19.5	23	19.5	34	19.5
			45-47	18	27-29	18	20-22	18	29-33	18
			44	17.5	30	17.5	19	17.5	28	17.5
			43	17	31	17	18	17	27	17
			40-42	16	32-39	16	16-17	16	21-26	16
			39	15	40	15	15	15	20	15
Tends to flush	40-59	Sometimes rotary seal is required	38	14.5	41	14.5	14	14.5	19	14.5
			34-37	12	42-49	12	11-13	12	11-18	12
			33	10	50	10	10	10	10	10
May flush	25-30	Rotary seal is necessary depending on flow speed and feeding conditions	32	9.5	51	9.5	9	9.5	9	9.5
			29-31	8	52-56	8	8	8	8	8
			28	6.25	57	6.25	7	6.25	7	6.25
Won't flush	0-24	Not required	27	6	58	6	6	6	6	6
			23-26	3	59-64	3	1-5	3	1-5	3
			>23	0	<64	0	0	0	0	0

Remarks : The above tables are reprinted from "Chemical Engineering" pages 166 and 167 of January 18,1965, with approval of Mr.Ralph Carr, Jr., and t
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2.7 Size distribution of particles

2.7.1 Arithmetic normal distributions

It is common practice to plot size distribution data in such a way that a straight line results. This can be done if the distribution fits a standard law, such as the normal law.

Although it might be expected that this type of distribution would be common, it seems to occur only for narrow size ranges of classified material.

An equation representing the normal distribution is

$$y = \frac{1}{\sigma\sqrt{(2\pi)}}\exp\left[-\frac{(x-\bar{x})^2}{2\sigma^2}\right] \qquad (2.15)$$

where $y = \frac{d\phi}{dx} = f(x)$

and $\int_{-\alpha}^{\alpha} f(x)dx = 1$ (i.e. the distribution is normalized)

σ is the standard deviation (σ^2 is the variance), $\bar{x}$ is the mean size and ϕ is a general term for the frequency, being number, length, surface or volume.

Let $t = \left(\frac{x-\bar{x}}{\sigma}\right)$

then $\sigma dt = dx$

equation 2.15 becomes

$$\frac{d\phi}{dt} = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{t^2}{2}\right)$$

Hence
$$\int_{0}^{\phi} d\phi = \frac{1}{2\pi} \int_{-\alpha}^{x} \exp\left(-\frac{t^2}{2}\right)^2 dt \qquad (2.16)$$

A plot of $\frac{d\phi}{dt}$ against t results in the well-know ‘dumb-bell’ shape of the normal probability curve **(Figure 2.10)**

Equation 2.16 is the basis for arithmetic probability graph paper and the integral is tabulated in any books on statistics **(Table 2.4)**

The integral from Table 2.5, for $t = 1$ is 84.13%, or $t = -1$ is 15.8%; therefore, the standard deviation

$$\sigma = \left(\frac{x - \bar{x}}{t}\right) = x_{84.13} - x_{50} = x_{50} - x_{15.87}$$

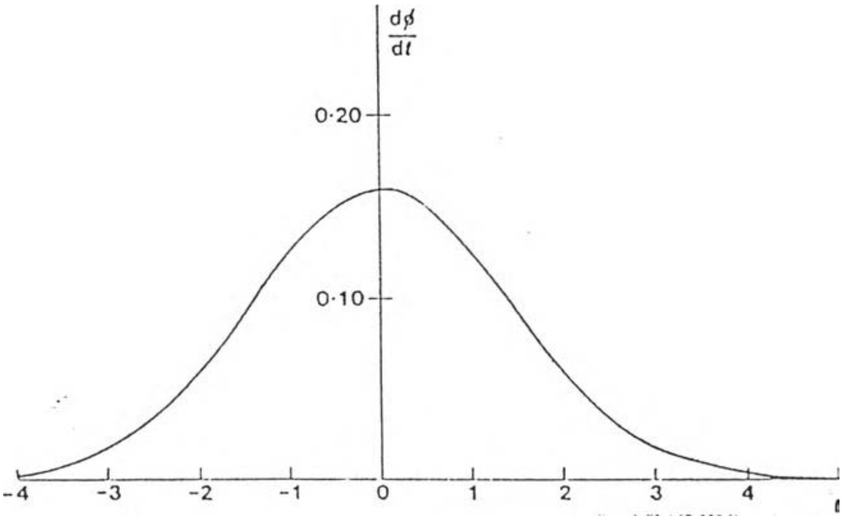


Figure 2.10 The normal probability curve. Relative frequency against standard deviation (68.26% of the distribution lies within 1 standard deviation $-1<t<+1$ of the mean.)

Table 2.4 Integration of the normal probability equation.

t	Integral
0	0.5000
0.5	0.6915
1.0	0.8413
1.5	0.9332
2	0.9772
3	0.9987
4	0.99997

2.7.2 The log-normal distribution

The equation of the log-normal distribution is obtained by replacing x with z = ln x, in equation 2.15. Then

$$y = \frac{1}{\sigma_z \sqrt{(2\pi)}} \exp\left[-\frac{(z-\bar{z})^2}{2\sigma_z{}^2} \right] \qquad (2.17)$$

where $y = \dfrac{d\phi}{d \ln x}$, σ_z is the standard deviation of z

and $$\bar{z} = \frac{\sum z d\phi}{\sum d\phi}$$

$$\bar{z} = \frac{\sum z d\phi}{\phi}$$

or $$\ln x_g = \frac{\sum \ln x d\phi}{\phi}$$

Therefore $$x_g = \left[\sqrt{\prod x^{d\phi}}\right]^{1/\phi} \qquad (2.18)$$

and $$\ln \sigma_g = \left[\frac{\sum n_i(\ln x_i - \ln \bar{x}_g)^2}{N-1}\right]^{1/2}$$

2.8 Analysis of Variance

The analysis of variance is a method for splitting the total variation of data into meaningful components that measure different sources of variation. One-Way Analysis of Variance is the simplest method. Assume that we are interested in the means $\mu_1, \mu_2, \ldots, \mu_k$ of k populations were taken into consideration whether they are different with some degree of significant. It will be assumed that the k populations are independent and normally distributed with common variance. We wish to derive appropriate methods for testing the hypothesis

$$H_0 \;:\; \mu_1 = \mu_2 = \ldots = \mu_k \qquad (2.19)$$

H_1 : at least two of the means are not equal (2.20)

Let x_{ij} denote the jth observation from the ith population and arrange the data as in **Table 2.5**. Here T_i is the total of all observations in the sample from the ith population, $\bar{x}_i$ the mean of all observation in the sample from the ith population, $T_{..}$ the total of all nk observations, and $\bar{x}_{..}$ the mean of all nk observations. Each observation may be written as

$$x_{ij} = \mu_i + \varepsilon_{ij}$$

where ε_{ij} is the deviation of the jth observation of the ith sample from the corresponding population mean. An alternative and preferred form of this equation is obtained by substituting $\mu_i = \mu+\alpha_i$, where μ is defined to be the mean of all the μ_i : that is,

$$\mu = \frac{\sum_{i=1}^{k}\mu_i}{k}$$

Hence we may write

$$x_{ij} = \mu + \alpha_i + \varepsilon_{ij},$$

subject to the restriction that $\sum_{i=1}^{k}\alpha_i = \sum_{i=1}^{k}(\mu_i - \mu) = 0$. It is customary to refer to α_i as the **effect** of the ith population.

The null hypothesis that the k population means are equal against the alternative that at least two of the means are unequal may also be stated by the equavalent hypothesis

H_0 : $\alpha_1 = \alpha_2 = \alpha_3 = 0$,

H_1 : At least one of the α_i is not equal to zero.

Our test will be based on a comparison of two independent estimates of the common population variance σ^2. These estimates will be obtained by splitting the total variability of our data into two components.

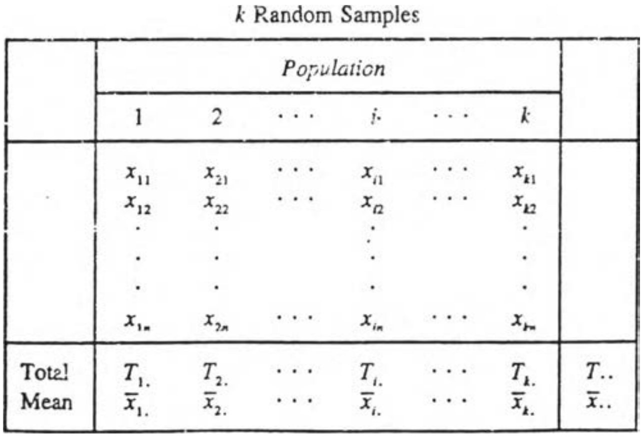
The variance of all the observations grouped into a single sample of size nk is given by the formula

$$s^2 = \frac{\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij} - \bar{x})^2}{nk - 1}$$

The double summation means that we sum all possible terms that are obtained by allowing i to assume values from 1 to k for each value of j from 1 to n. The numerator of s^2, called the **total sum of squares**, measures the total variability of our data. It may be partitioned by means of the following identity.

$$\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij}-\overline{x}_{..})^2 = n\sum_{i=1}^{k}(\overline{x}_{i.}-\overline{x}_{..})^2 + \sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij}-\overline{x}_{i.})^2$$

Table 2.5 k Random Samples



k Random Samples

	Population						
	1	2	···	*i*	···	*k*	
	x_{11}	x_{21}	···	x_{i1}	···	x_{k1}	
	x_{12}	x_{22}	···	x_{i2}	···	x_{k2}	
	·	·		·		·	
	·	·		·		·	
	·	·		·		·	
	x_{1n}	x_{2n}	···	x_{in}	···	x_{kn}	
Total	$T_{1.}$	$T_{2.}$	···	$T_{i.}$	···	$T_{k.}$	$T_{..}$
Mean	$\overline{x}_{1.}$	$\overline{x}_{2.}$	···	$\overline{x}_{i.}$	···	$\overline{x}_{k.}$	$\overline{x}_{..}$

It will be convenient in what follows to identify the terms of the sum-of-squares identity by the following notation:

$$SST = \sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij}-\overline{x}_{..})^2 = \text{total sum of squares,}$$

$$SSC = n\sum_{i=1}^{k}(x_{i}-\overline{x}_{..})^2 = \text{sum of squares for column means,}$$

$$SSE = \sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij}-\overline{x})^2 = \text{error sum of squares.}$$

The sum-of-squares identity can then be represented symbolically by the equation

$$SST = SSC + SSE.$$

Many theories refer to the sum of squares for column means as the treatment sum of squares. This terminology is derived from the fact that the k different populations are often classified according to different treatments. Thus the observations x_{ij} (j = 1, 2, ..., n) represent the n measurements corresponding to the ith treatment. Today the term "treatment" is used more generally to refer to the various classifications, whether they be different analysis, different fertilizers, different manufacturers, or different regions of the country.

One estimate of σ^2, based on k-1 degrees of freedom, is given by

$$s_1^2 = \frac{SSC}{k-1}$$

If H_0 is true, s_1^2 is an unbiased estimate of σ^2. However, if H_1 is true, SSC will tend to have a large numerical value and s_1^2 will likely overestimate σ^2. A second and independent estimate of σ^2, based on k(n-1) degrees of freedom, is given by

$$s_2^2 = \frac{SSE}{k(n-1)}$$

The estimate s_2^2 is unbiased regardless of the truth or falsity of the null hypothesis. We have already seen that the variance of our grouped data, with *nk*-1 degrees of freedom, is

$$s^2 = \frac{SST}{nk-1}$$

which is an unbiased estimate of σ^2 when H_0 is true. It is important to note that the sum-of-squares identity has not only partitioned the total variability of the data but also the total number of degrees of freedom; that is,

$$nk-1 = k-1 + k(n-1)$$

When H_0 is true, the ratio

$$F = \frac{s_1^2}{s_2^2}$$

is a value of the random variable F having the F distribution with k-1 and k(n-1) degrees of freedom. Since s_1^2 overestimates σ^2 when H_0 is false, we have a one-tailed test with the critical region entirely in the right tail of the distribution. The null hypothesis H_0 is rejected at the α level of significance when

$$F > F_\alpha[k-1, k(n-1)]$$

The previously defined formulas for SST, SSC and SSE are not in the best computationed form. In actual practice, one computes SST and SSC by the following equivalent formulas and then, making use of the sum-of-squares identity, obtains SSE by subtraction.

Sum-of-Squares Computational Formulas

$$SST = \sum_{i=1}^{k}\sum_{j=1}^{n} x_{ij}^2 - \frac{T_{..}^2}{nk},$$

$$SSC = \frac{\sum_{i=1}^{k} T_{i.}^2}{n} - \frac{T_{..}^2}{nk}$$

$$SSE = SST\text{-}SSC$$

The computations in an analysis-of-variance problem are usually summarized in tabular form, as shown in **Table 2.6**. It is customary to refer to the various estimates of σ^2 as the **mean squares.**

Table 2.6 Analysis of Variance for the One-Way Classification.

Source of Variation	*Sum of Squares*	*Degrees of Freedom*	*Mean Square*	*Computed f*
Column means	SSC	$k-1$	$s_1^2 = \frac{SSC}{k-1}$	$\frac{s_1^2}{s_2^2}$
Error	SSE	$k(n-1)$	$s_2^2 = \frac{SSE}{k(n-1)}$	
Total	SST	$nk-1$		