Chapter II

LINEAR SYSTEM IDENTIFICATION BY CROSS-CORRELATION TECHNIQUE

2.1 Introduction

In the study of a linear system identification by cross-correlation technique, the output response at time t may be expressed in term of all previous inputs through a convolution integral equation

$$y(t) = \int_{0}^{\infty} g(\tau)x(t-\tau)d\tau \qquad (1)$$

where g(t) is an impulse response of the linear system, and g(t) = 0, $t < 0$.

The cross-correlation function between the output, y(t), and the input, x(t), may be written in the form

$$\phi_{xy}(\tau) = \int_{-\infty}^{\infty} g(s)\phi_{xx}(\tau-s)ds \qquad (2)$$

where $\phi_{xx}(\tau)$ is the autocorrelation function of the input signal. If the input signal is white noise, its autocorrelation function $\phi_{xx}(\tau)$ is approximately a delta function. Then the impulse response function can be directly obtained by

$$g(\tau) = K_g \phi_{xy}(\tau) \qquad (3).$$

However, it has two practical disadvantages[14] :

(a) A very long correlation time is necessary to ensure that $\phi_{xx}(\tau)$ approximates to a delta function,

(b) Practical difficulties can arise in transmitting an amplitude modulated signal without distortion, especially where electromechanical transducers are invoIed.

These disadvantages may be overcome by using pseudo-random binary maximum length sequences[8,9,10].

2.2 Pseudo-Random Binary Maximum Length Sequences

The most widely used pseudo-random noise is the binary maximum length sequence[11,12]. The autocorrelation of the b.m.l.s. is similar to those of white noise. Morever, the b.m.l.s. posses other properties[5,6,8,14,20] which are useful for the calculation of the cross-correlation function, $\phi_{xy}(\tau)$.

2.3 Autocorrelation Function of Binary Maximum Length Sequences

Consider a b.m.l.s., x(t), with amplitude of ±a and the time period T shown in Fig. 1. The autocorrelation function $\phi_{xx}(\tau)$ is

$$\phi_{xx}(\tau) = \frac{1}{T}\int_{0}^{T} x(t)x(t+\tau)\,dt \qquad (4).$$

The wave form of $\phi_{xx}(\tau)$ is illustrated in Fig. 2.

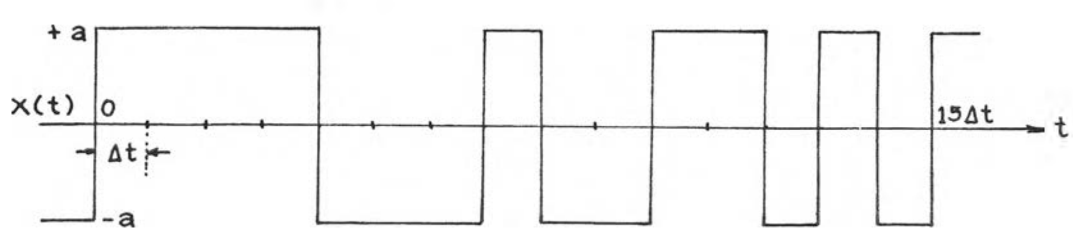

Fig. 1 A b.m.l.s. with the period of 15 bits.

Consider a b.m.l.s. generated by n states shift register generator. Its time period is $N\Delta t$, where N is the number of bits in one period and equal to 2^n-1, and Δt is the constant time-bit interval which is the

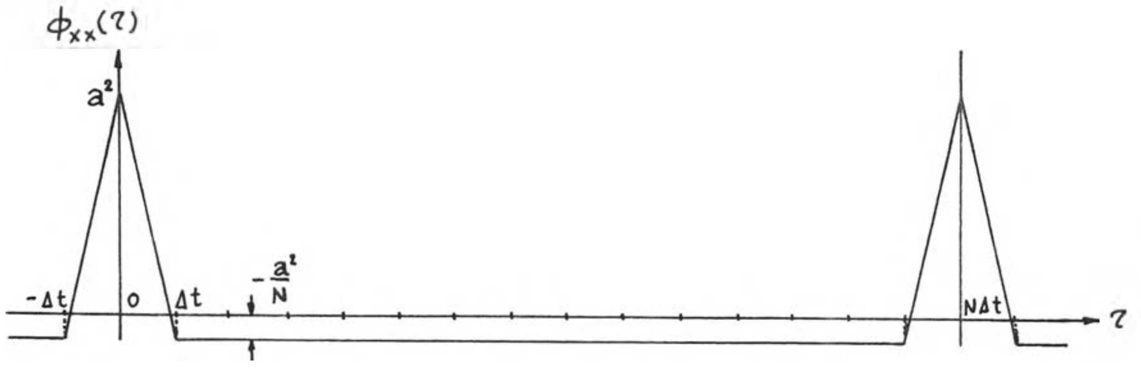

Fig. 2 Continuous autocorrelation function of b.m.l.s. illustrated in Fig. 1

clock pulse period[5]. In discrete form, the autocorrelation function may be written as

$$\phi_{xx}(i\Delta t) = \frac{1}{N}\sum_{j=0}^{N-1} x(j\Delta t)x(j\Delta t+i\Delta t) \qquad (5)$$

where i is an integer corresponding to a shift from the original sequence.

It has been shown that the autocorrelation function[5] is

$$\begin{aligned} \phi_{xx}(i\Delta t) &= a^2 && \text{for } i = 0, N, 2N, \ldots\ldots \\ &= -\frac{a^2}{N} && \text{otherwise} \end{aligned} \qquad (6).$$

The discrete wave form of $\phi_{xx}(i\Delta t)$ is shown in Fig. 3.

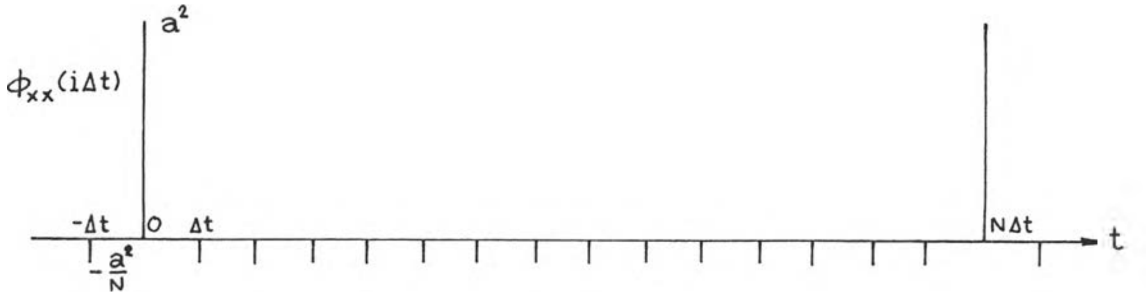

Fig. 3 Discrete autocorrelation function of b.m.l.s.

For the intermediate value discrete form[17], the autocorrelation function of Eqn. (6) can be illustrated in Fig. 4.

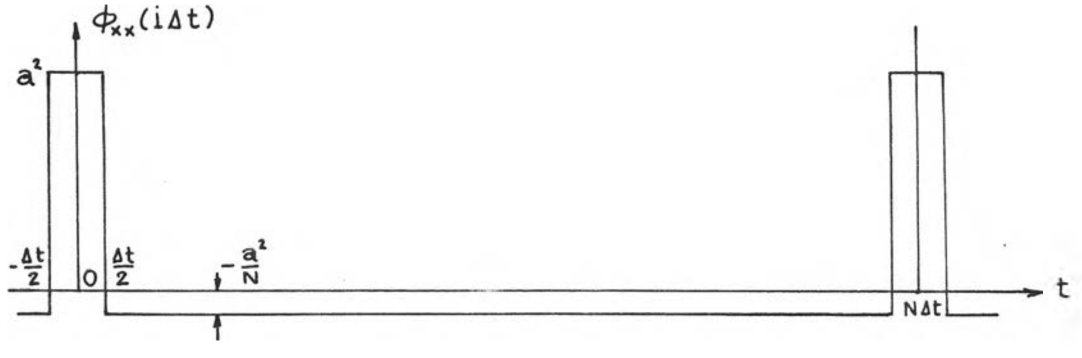

Fig. 4 Intermediate value discrete form of autocorrelation function of b.m.l.s.

Suppose that the time-bit interval, Δt, is divided into m small intervals. For the small interval $\Delta t/m$ as the interval of integration, the autocorrelation function in Eqn. (5) becomes

$$\phi_{xx}(i\Delta t+\frac{\ell}{m}\Delta t) = \frac{1}{N}\sum_{j=0}^{N-1}\left\{\frac{1}{m}\sum_{k=0}^{m-1}x(j\Delta t+\frac{k}{m}\Delta t)x(j\Delta t+\frac{k}{m}\Delta t+i\Delta t+\frac{\ell}{m}\Delta t)\right\}$$

$$= \frac{1}{mN}\sum_{j=0}^{N-1}\sum_{k=0}^{m-1}x(j\Delta t+\frac{k}{m}\Delta t)x(j\Delta t+\frac{k}{m}\Delta t+i\Delta t+\frac{\ell}{m}\Delta t) \qquad (7)$$

where ℓ is an integer, $0 \leq \ell \leq m-1$. It has been shown that the autocorrelation function[2] in Eqn. (7) may be written as

$$\phi_{xx}(i\Delta t+\frac{\ell}{m}\Delta t) = \frac{a^2(N+1)}{N}\left\{1-\frac{|r|}{m}\right\}-\frac{a^2}{N} \qquad \text{for } i = 0,N,2N,\ldots$$

$$= -\frac{a^2}{N} \qquad \text{otherwise} \qquad (8)$$

where r is an integer, $-(m-1) \leq r \leq m-1$.

When the value of m approaches infinity, Eqn. (8) degenerates into the result obtained by the continuous integration.

In addition, if the time-bit interval, Δt, is divided into m small intervals, and the interval of integration is Δt [13,19], the auto-correlation function is

$$\phi_{xx}(i\Delta t+\frac{\ell}{m}\Delta t) = \frac{1}{N}\sum_{j=0}^{N-1} x(j\Delta t)x(j\Delta t+i\Delta t+\frac{\ell}{m}\Delta t) \qquad (9)$$

where ℓ is an integer, $0 \leq \ell \leq m-1$.

Since the value of ℓ/m is less than 1, and the magnitude of the b.m.l.s. input, x(t), within the time-bit interval remains the same. Then Eqn. (9) reduces to

$$\phi_{xx}(i\Delta t+\frac{\ell}{m}\Delta t) = \frac{1}{N}\sum_{j=0}^{N-1} x(j\Delta t)x(j\Delta t+i\Delta t) \qquad (10).$$

From Eqns. (5) and (6), we obtain

$$\begin{aligned}\phi_{xx}(i\Delta t+\frac{\ell}{m}\Delta t) &= a^2 && \text{for } i = 0,N,2N,\ldots.\\ &= -\frac{a^2}{N} && \text{otherwise} \qquad (11).\end{aligned}$$

However, when the value of m approaches infinity, Eqn. (11) yields the continuous autocorrelation function of the form shown in Fig. 5.

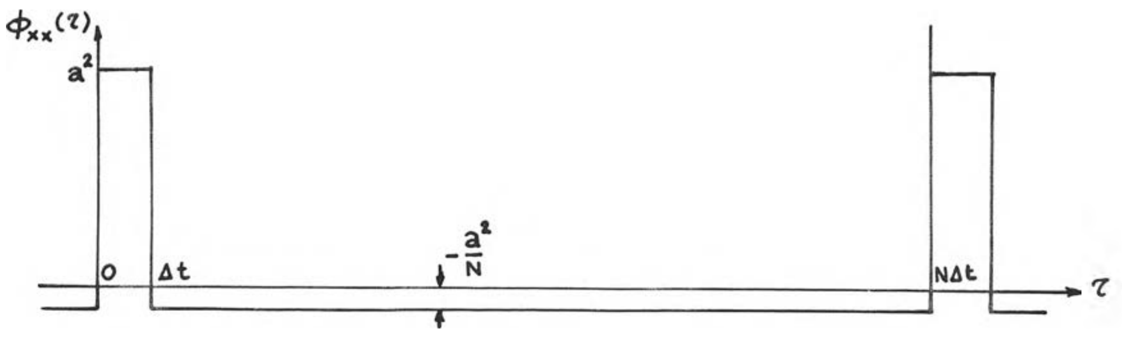

Fig. 5 The autocorrelation function of b.m.l.s. satisfying Eqn. (11) when m approaches infinity

The integration method using Δt as the interval of integration is proposed for a new method of cross-correlation technique which will be described in Section 2.5.

2.4 Discrete Cross-Correlation Technique

For the discrete identification of a linear system with a b.m.l.s. as an input signal, the system output is sampled at time $t = (i+\lambda)\Delta t$, where $i = 0,1,2,\ldots$ and the value of λ is in the range $0 \leq \lambda \leq 1$. The output sequence can be expressed by

$$y[(i+\lambda)\Delta t] = \Delta t \sum_{j=0}^{\infty} g(j\Delta t)x[(i+\lambda-j)\Delta t] \qquad (12).$$

The cross-correlation function between the discrete input and the discrete output is

$$\phi_{xy}(i\Delta t) = \frac{1}{N}\sum_{j=0}^{N-1} x[(j+\lambda)\Delta t]\, y[(i+j+\lambda)\Delta t] \qquad (13).$$

From Eqns. (12) and (13), the convolution integral can be written as

$$\phi_{xy}(i\Delta t) = \Delta t \sum_{j=0}^{\infty} g(j\Delta t)\phi_{xx}[(i-j)\Delta t] \qquad (14).$$

For a physically realizable system and the impulse response decays to zero within the time period of the input sequence, $N\Delta t$, the cross-correlation $\phi_{xy}(i\Delta t)$ in Eqn. (14) reduces to

$$\phi_{xy}(i\Delta t) = \Delta t \sum_{j=0}^{N-1} g(j\Delta t)\phi_{xx}[(i-j)\Delta t] \qquad (15).$$

Since the input signal is a b.m.l.s. and from Eqn. (6), the equation (15) becomes

$$\phi_{xy}(i\Delta t) = \frac{a^2(N+1)}{N}\Delta t \sum_{j=0}^{N-1} g(j\Delta t)\, \delta_r[(i-j)\Delta t] - \frac{a^2}{N}\Delta t \sum_{j=0}^{N-1} g(j\Delta t) \qquad (16)$$

where $\delta_r(i\Delta t)$ is a unit rectangular pulse of pulse width Δt, and $\delta_r(i\Delta t)$ may be expressed in the form

$$\delta_r(i\Delta t) = u(i\Delta t+\lambda\Delta t)-u(i\Delta t+\lambda\Delta t-\Delta t) \qquad (17).$$

Consider the expression $\sum_{j=0}^{N-1} g(j\Delta t)\delta_r[(i-j)\Delta t]$ in Eqn. (16) and the value of $\delta_r(i\Delta t)$ in Eqn. (17), we may write[13]

$$\sum_{j=0}^{N-1} g(j\Delta t)\delta_r[(i-j)\Delta t] = \frac{1}{\Delta t}\Big[S[(i+\lambda)\Delta t]-S(0)\Big] \qquad \text{for } i = 0$$

$$= \frac{1}{\Delta t}\Big[S[(i+\lambda)\Delta t]-S[(i+\lambda-1)\Delta t]\Big] \qquad \text{for } i \neq 0 \qquad (18)$$

where $S(\tau)$ is the step response of the system.

By Taylor's series expansions of $S[(i+\lambda)\Delta t]$ and $S[(i+\lambda-1)\Delta t]$ about $S(\tau)$, where $\tau = (i+\lambda-\mu)\Delta t$ and the value of μ is in the range $0 \leq \mu \leq 1$, we have

$$S[(i+\lambda)\Delta t] = S(\tau)+\sum_{k=1}^{\infty}\frac{\mu^k(\Delta t)^k}{k!}S^{(k)}(\tau) \qquad (19)$$

$$S[(i+\lambda-1)\Delta t] = S(\tau)+\sum_{k=1}^{\infty}\frac{(\mu-1)^k(\Delta t)^k}{k!}S^{(k)}(\tau) \qquad (20)$$

For the case $i \neq 0$, Eqn. (18) can be written as

$$\sum_{j=0}^{N-1} g(j\Delta t)\delta_r[(i-j)\Delta t] = \frac{1}{\Delta t}\sum_{k=1}^{\infty}\frac{(\Delta t)^k}{k!}\Big[\mu^k-(\mu-1)^k\Big]S^{(k)}(\tau)$$

$$= S'(\tau)+\sum_{k=2}^{\infty}\frac{(\Delta t)^{k-1}}{k!}\Big[\mu^k-(\mu-1)^k\Big]S^{(k)}(\tau) \qquad (21).$$

Since $g(t) = \frac{dS(t)}{dt}$, Eqn. (21) becomes

$$\sum_{j=0}^{N-1} g(j\Delta t)\delta_r[(i-j)\Delta t] = g(\tau)+\sum_{j=1}^{\infty}\frac{(\Delta t)^j}{(j+1)!}\Big[\mu^{j+1}-(\mu-1)^{j+1}\Big]S^{(j)}(\tau) \qquad (22).$$

Similarly for the case $i = 0$, Eqn. (18) can be rewritten as

$$\sum_{j=0}^{N-1} g(j\Delta t)\,\delta_r[(i-j)\Delta t] = \lambda g(\tau)+\sum_{j=1}^{\infty} \frac{(\Delta t)^j}{(j+1)!}\left[\mu^{j+1}-(\mu-\lambda)^{j+1}\right] s^{(j)}(\tau) \qquad (23).$$

It can be seen that the odd terms of the series representing the error in estimating $g(\tau)$ in the right hand side of Eqn. (22) are zero when $\mu = 1/2$. This value of μ also minimises the factor $\mu^{j+1}-(\mu-1)^{j+1}$ in the even terms. The impulse response is obtained at discrete time $i\Delta t$ when $\lambda = \mu = 1/2$ and the output must be sampled at the time $t = (i+1/2)\Delta t$. The sampling at the middle of the time-bit interval of a signal is called the intermediate value sequence.

Substituting Eqns. (22) and (23) into Eqn. (16) and for $\lambda = \mu = 1/2$, we obtain

$$\phi_{xy}(i\Delta t) = \frac{a^2(N+1)}{2N}\Delta t\left[g(i\Delta t)+\sum_{j=1}^{\infty}\frac{(\Delta t)^j}{2^j(j+1)!}\,g^{(j)}(i\Delta t)\right]-\frac{a^2}{N}\Delta t\sum_{j=0}^{N-1} g(j\Delta t) \qquad \text{for } i = 0$$

$$= \frac{a^2(N+1)}{N}\Delta t\left[g(i\Delta t)+\sum_{j=1}^{\infty}\frac{(\Delta t)^{2j}}{2^{2j}(2j+1)!}\,g^{(2j)}(i\Delta t)\right]-\frac{a^2}{N}\Delta t\sum_{j=0}^{N-1} g(j\Delta t) \qquad \text{for } i \neq 0 \qquad (24).$$

This is the most accurate estimate of $g(i\Delta t)$ which can be achieved by this discrete cross-correlation method. The same coefficient of the derivative terms can be obtained by continuous integration if the auto-correlation of the input signal is considered to be a unit rectangular pulse as shown in Appendix A.

If sample sizes are increased and suppose that the system output $y(t)$ is sampled at the time $t = (i+1/2)\Delta t/m$, where m is an integer and $m > 0$, then the cross-correlation function between the discrete input and the discrete output is performed over mN samples with $\Delta t/m$ as the interval of integration. We have

$$\phi_{xy}(\frac{i}{m}\Delta t) = \frac{1}{m}\Delta t \sum_{j=0}^{mN-1} g(\frac{j}{m}\Delta t)\phi_{xx}(\frac{i-j}{m}\Delta t) \tag{25}$$

From Eqn. (8), the equation (25) may be rewritten as

$$\phi_{xy}(\frac{i}{m}\Delta t) = \frac{a^2(N+1)}{mN}\Delta t \sum_{j=0}^{mN-1} g(\frac{j}{m}\Delta t)\delta_t(\frac{i-j}{m}\Delta t) - \frac{a^2}{mN}\Delta t \sum_{j=0}^{mN-1} g(\frac{j}{m}\Delta t). \tag{26}$$

where $\delta_t(\frac{r}{m}\Delta t)$ is the discrete representation of the unit triangular pulse of pulse width $2\Delta t$. The function $\delta_t(\frac{r}{m}\Delta t)$ can be expressed as

$$\delta_t(\frac{r}{m}\Delta t) = \{1-\frac{|r|}{m}\}\delta_r(\frac{r}{m}\Delta t) \qquad \text{for } -(m-1) \leqslant r \leqslant m-1$$

$$= 0 \qquad \text{otherwise} \tag{27}$$

where $\delta_r(\frac{r}{m}\Delta t)$ is the unit rectangular pulse of pulse width $\Delta t/m$ and $\delta_r(\frac{r}{m}\Delta t)$ may be expressed in the form

$$\delta_r(\frac{r}{m}\Delta t) = u(\frac{r+\lambda}{m}\Delta t)-u(\frac{r+\lambda-1}{m}\Delta t) \tag{28}$$

Substituting Eqn. (27) into Eqn. (26) and from Eqn. (28) for the value of $\lambda = 1/2$, we obtain

$$\phi_{xy}(\frac{i}{m}\Delta t) = \frac{a^2(N+1)}{mN}\Delta t\Big[\sum_{r=1}^{m-1}(1-\frac{r}{m})(\frac{m}{\Delta t})\{S(\frac{r+1/2}{m}\Delta t)-S(\frac{r-1/2}{m}\Delta t)\}+\frac{m}{\Delta t}\{S(\frac{1}{2m}\Delta t)$$

$$-S(0)\}\Big]-\frac{a^2}{mN}\Delta t\sum_{j=0}^{mN-1} g(\frac{j}{m}\Delta t) \qquad \text{for } i = 0$$

$$= \frac{a^2(N+1)}{mN}\Delta t \sum_{r=-(m-1)}^{m-1} (1-\frac{|r|}{m})(\frac{m}{\Delta t})\{S(\frac{i+r+1/2}{m}\Delta t)-S(\frac{i+r-1/2}{m}\Delta t)\}$$

$$-\frac{a^2}{mN}\Delta t \sum_{j=0}^{mN-1} g(\frac{j}{m}\Delta t) \qquad \text{for } i \geqslant m$$

For the case $i \geqslant m$, by Taylor's Series expansions and $g(t) = \frac{dS(t)}{dt}$, we can write

$$\frac{1}{m}\sum_{r=-(m-1)}^{m-1}(1-\frac{|r|}{m})(\frac{m}{\Delta t})\left[S(\frac{i+r+1/2}{m}\Delta t)-S(\frac{i+r-1/2}{m}\Delta t)\right]$$

$$=\frac{1}{m}\sum_{r=-(m-1)}^{m-1}(1-\frac{|r|}{m})\left[g(\frac{i}{m}\Delta t)+\sum_{j=1}^{\infty}\frac{(\Delta t)^j}{m^j(j+1)!}\{(r+\frac{1}{2})^{j+1}\right.$$

$$\left.-(r-\frac{1}{2})^{j+1}\}g^{(j)}(\frac{i}{m}\Delta t)\right]$$

$$=\frac{1}{m}g(\frac{i}{m}\Delta t)\sum_{r=-(m-1)}^{m-1}(1-\frac{|r|}{m})+\sum_{j=1}^{\infty}\frac{(\Delta t)^j}{m^{j+1}(j+1)!}\left[\sum_{r=-(m-1)}^{m-1}(1-\frac{|r|}{m})\{(r+\frac{1}{2})^{j+1}\right.$$

$$\left.-(r-\frac{1}{2})^{j+1}\}\right]g^{(j)}(\frac{i}{m}\Delta t)$$

$$=g(\frac{i}{m}\Delta t)+\sum_{j=1}^{\infty}\frac{(\Delta t)^j}{m^{j+1}(j+1)!}\left[\sum_{r=-(m-1)}^{m-1}(1-\frac{|r|}{m})\{(r+\frac{1}{2})^{j+1}\right.$$

$$\left.-(r-\frac{1}{2})^{j+1}\}\right]g^{(j)}(\frac{i}{m}\Delta t) \qquad (30)$$

Since the expression $\sum_{r=-(m-1)}^{m-1}(1-\frac{|r|}{m})\{(r+\frac{1}{2})^{j+1}-(r-\frac{1}{2})^{j+1}\}$ is zero for the odd value of j, then Eqn. (30) becomes

$$\frac{1}{m}\sum_{r=-(m-1)}^{m-1}(1-\frac{|r|}{m})(\frac{m}{\Delta t})\left[S(\frac{i+r+1/2}{m}\Delta t)-S(\frac{i+r-1/2}{m}\Delta t)\right]$$

$$=g(\frac{i}{m}\Delta t)+\sum_{j=1}^{\infty}\frac{(\Delta t)^{2j}}{m^{2j+1}(2j+1)!}\left[\sum_{r=-(m-1)}^{m-1}(1-\frac{|r|}{m})\{(r+\frac{1}{2})^{2j+1}\right.$$

$$\left.-(r-\frac{1}{2})^{2j+1}\}\right]g^{(2j)}(\frac{i}{m}\Delta t)$$

$$=g(\frac{i}{m}\Delta t)+\sum_{j=1}^{\infty}\frac{(\Delta t)^{2j}}{m^{2j+1}(2j+1)!}\left[2\sum_{r=0}^{m-1}(1-\frac{|r|}{m})\{(r+\frac{1}{2})^{2j+1}-(r-\frac{1}{2})^{2j+1}\}\right.$$

$$\left.-2(\frac{1}{2})^{2j+1}\right]g^{(2j)}(\frac{i}{m}\Delta t) \qquad (31)$$

Eqn. (31) can be reduced to

$$\frac{1}{m}\sum_{r=-(m-1)}^{m-1}(1-\frac{|r|}{m})(\frac{m}{\Delta t})\left[S(\frac{i+r+1/2}{m}\Delta t)-S(\frac{i+r-1/2}{m}\Delta t)\right]$$

$$=g(\frac{i}{m}\Delta t)+\sum_{j=1}^{\infty}\left[\frac{(\Delta t)^{2j}}{2^{2j}(2j+1)!\,m^{2j+2}}\sum_{r=1}^{m}(2r-1)^{2j+1}g^{(2j)}(\frac{i}{m}\Delta t)\right] \qquad (32)$$

For the case i = 0, we also obtain

$$\frac{1}{m}\sum_{r=-(m-1)}^{m-1}(1-\frac{|r|}{m})(\frac{m}{\Delta t})\left[S(\frac{i+r+1/2}{m}\Delta t)-S(0)\right]$$

$$=\frac{1}{2}g(\frac{i}{m}\Delta t)+\sum_{j=1}^{\infty}\left[\frac{(\Delta t)^j}{2^{j+1}(j+1)!\,m^{j+2}}\sum_{r=1}^{m}(2r-1)^{j+1}g^{(j)}(\frac{i}{m}\Delta t)\right] \tag{33}$$

Substituting Eqns. (32) and (33) into Eqn. (29), the cross-correlation is obtained as

$$\phi_{xy}(\frac{i}{m}\Delta t) = \frac{a^2(N+1)}{2N}\Delta t\left[g(\frac{i}{m}\Delta t)+\sum_{j=1}^{\infty}\{\frac{(\Delta t)^j}{2^j(j+1)!\,m^{j+2}}\sum_{r=1}^{m}(2r-1)^{j+1}g^{(j)}(\frac{i}{m}\Delta t)\}\right]$$

$$-\frac{a^2}{mN}\Delta t\sum_{j=0}^{mN-1}g(\frac{j}{m}\Delta t) \qquad \text{for } i = 0$$

$$= \frac{a^2(N+1)}{N}\Delta t\left[g(\frac{i}{m}\Delta t)+\sum_{j=1}^{\infty}\{\frac{(\Delta t)^{2j}}{2^{2j}(2j+1)!\,m^{2j+2}}\sum_{r=1}^{m}(2r-1)^{2j+1}g^{(2j)}(\frac{i}{m}\Delta t)\}\right]$$

$$-\frac{a^2}{mN}\Delta t\sum_{j=0}^{mN-1}g(\frac{j}{m}\Delta t) \qquad \text{for } i \geqslant m \tag{34}$$

When m = 1, Eqn. (34) can be reduced to Eqn. (24). When the value of $m > 1$, the coefficient of each derivative term in Eqn.(34) is greater than the coefficient of the corresponding derivative term in Eqn. (24). Morever, when m approaches infinity, Eqn. (34) yields the result which is identical to those of the continuous cross-correlation function obtained by using correlator. Thus, the discrete cross-correlation, Eqn. (24), provides a more accurate estimate of system impulse response than that obtained by continuous cross-correlation.

From Eqn. (24), the impulse response can be evaluated as

$$g(i\Delta t) = \frac{2N}{a^2(N+1)\Delta t}\left[\phi_{xy}(0)+\frac{a^2}{N}\Delta t\sum_{j=0}^{N-1}g(j\Delta t)\right]-\sum_{j=1}^{\infty}\frac{(\Delta t)^j}{2^j(j+1)!}g^{(j)}(0)$$

$$\text{for } i = 0$$

$$g(i\Delta t) = \frac{N}{a^2(N+1)\Delta t}\left[\phi_{xy}(i\Delta t)+\frac{a^2}{N}\Delta t\sum_{j=0}^{N-1} g(j\Delta t)\right]-\sum_{j=1}^{\infty}\frac{(\Delta t)^{2j}}{2^{2j}(2j+1)!}g^{(2j)}(i\Delta t)$$

otherwise (35)

This process of the discrete linear system identification will be used in Chapter 4.

2.5 New Method of Correlation Technique for Identification

For this new method, the system output y(t) will be sampled at time $t = i\Delta t+(\ell+1/2)\Delta t/m$, where $i = 0,1,2,....$ and m is the number of sampled points in time-bit interval, Δt, and $\ell = 0,1,2,....,(m-1)$. The output sequence $\{y_j\}$ obtained by this method is the intermediate value sequence. If the b.m.l.s. input signal is sampled at time $t = i\Delta t +(\ell+1/2)\Delta t/m$, then the cross-correlation between the output sequence and the input sequence with Δt as the interval of integration can be expressed as

$$\phi_{xy}(i\Delta t+\frac{\ell}{m}\Delta t) = \frac{1}{m}\Delta t\sum_{j=0}^{mN-1} g(\frac{j}{m}\Delta t)\phi_{xx}(i\Delta t+\frac{\ell}{m}\Delta t-\frac{j}{m}\Delta t) \qquad (36)$$

where $\phi_{xx}(i\Delta t+\frac{\ell}{m}\Delta t)$ is the autocorrelation function of the input sequence expressed by Eqn. (11).

Then, we obtain

$$\phi_{xy}(i\Delta t+\frac{\ell}{m}\Delta t) = \frac{a^2(N+1)}{mN}\Delta t\sum_{j=0}^{mN-1} g(\frac{j}{m}\Delta t)\delta_r(i\Delta t+\frac{\ell}{m}\Delta t-\frac{j}{m}\Delta t)$$
$$-\frac{a^2}{mN}\Delta t\sum_{j=0}^{N-1}\sum_{r=0}^{m-1} g(j\Delta t+\frac{r}{m}\Delta t) \qquad (37)$$

where $\delta_r(i\Delta t)$ is a unit rectangular pulse of pulse width Δt. This rectangular pulse may be written in the sequence of the form as $\{1_1,1_2,1_3,....,1_{m-1}\}$ which is the train of unit rectangular pulses of pulse width $\Delta t/m$.

Applying the similar technique described in Section 2.4, Eqn. (37) may be rewritten as

$$\phi_{xy}(i\Delta t+\frac{\ell}{m}\Delta t) = \frac{a^2(N+1)}{mN}\Delta t\left[\sum_{r=0}^{\ell} g_e(\frac{r}{m}\Delta t)-\frac{1}{2}g_e(0)\right]-\frac{a^2}{mN}\Delta t\sum_{j=0}^{N-1}\sum_{r=0}^{m-1} g(j\Delta t+\frac{r}{m}\Delta t) \quad \text{for } i = 0$$

$$= \frac{a^2(N+1)}{mN}\Delta t\sum_{r=0}^{m-1} g_e(i\Delta t+\frac{r}{m}\Delta t)-\frac{a^2}{mN}\Delta t\sum_{j=0}^{N-1}\sum_{r=0}^{m-1} g(j\Delta t+\frac{r}{m}\Delta t) \quad \text{for } i \neq 0 \qquad (38)$$

where $g_e(i\Delta t+\frac{r}{m}\Delta t)$ is the system impulse response function with discrete interval $\Delta t/m$, including the error due to the derivative terms. The value of $g_e(i\Delta t+\frac{r}{m}\Delta t)$ can be expressed by

$$g_e(i\Delta t+\frac{r}{m}\Delta t) = g(0)+\sum_{j=1}^{\infty}\frac{(\Delta t/m)^j}{2^j(j+1)!}g^{(j)}(0) \quad \text{for } i = r = 0$$

$$=g(i\Delta t+\frac{r}{m}\Delta t)+\sum_{j=1}^{\infty}\frac{(\Delta t/m)^{2j}}{2^{2j}(2j+1)!}g^{(2j)}(i\Delta t+\frac{r}{m}\Delta t) \quad \text{otherwise} \qquad (39)$$

From Eqn. (38), we obtain

$$g_e(0) = \frac{2mN}{a^2(N+1)\Delta t}\left[\phi_{xy}(0)+\frac{a^2}{mN}\Delta t\sum_{j=0}^{N-1}\sum_{r=0}^{m-1} g(j\Delta t+\frac{r}{m}\Delta t)\right] \qquad (40)$$

$$g_e(\frac{\ell}{m}\Delta t) = \frac{mN}{a^2(N+1)\Delta t}\left[\phi_{xy}(\frac{\ell}{m}\Delta t)-\phi_{xy}(\frac{\ell-1}{m}\Delta t)\right] \quad \text{for } \ell = 1,2,\ldots,m-1 \qquad (41)$$

$$g_e(\Delta t) = \frac{mN}{a^2(N+1)\Delta t}\left[\phi_{xy}(\Delta t)-\phi_{xy}(\frac{m-1}{m}\Delta t)\right]+\frac{1}{2}g_e(0) \qquad (42)$$

$$g_e(i\Delta t+\frac{\ell}{m}\Delta t) = \frac{mN}{a^2(N+1)\Delta t}\left[\phi_{xy}(i\Delta t+\frac{\ell}{m}\Delta t)-\phi_{xy}(i\Delta t+\frac{\ell-1}{m}\Delta t)\right]$$

$$+g_e(i\Delta t-\Delta t+\frac{\ell}{m}\Delta t) \quad \text{otherwise} \qquad (43)$$

The linear system impulse response function at discrete time $t = i\Delta t+\ell\Delta t/m$, where $i = 0,1,2,\ldots$, $\ell = 0,1,2,\ldots,(m-1)$, and m is the number of sampled points in time-bit interval, Δt, can be determined by using Eqns. (39), (40), (41), (42), and (43). This method provides additional discrete points of the system impulse response function by using the shorter period of the b.m.l.s. input signal.