CHAPTER I

PRELIMINARIES

In this chapter we shall give some notations, definitions and theorems used in this thesis. Our notations are:

$\mathbb{Z}$ is the set of all integers,

$\mathbb{Z}^+$ is the set of all positive integers,

$\mathbb{Z}^+_0 = \mathbb{Z}^+ \cup \{0\}$

$\mathbb{Q}$ is the set of all rational numbers,

$\mathbb{Q}^+$ is the set of all positive rational numbers,

$\mathbb{Q}^+_0 = \mathbb{Q}^+ \cup \{0\}$

$\mathbb{R}$ is the set of all real numbers,

$\mathbb{R}^+$ is the set of all positive real numbers,

$\mathbb{R}^+_0 = \mathbb{R}^+ \cup \{0\}$

$(\mathbb{Z}^+,\min,\cdot),(\mathbb{Q}^+,\min,\cdot),(\mathbb{R}^+,\min,\cdot)$ we mean that $\mathbb{Z}^+,\mathbb{Q}^+, \mathbb{R}^+$ have the usual multiplication and $x+y = \min\{x,y\}$ (minimum of x,y)

$(\mathbb{Z}^+,\max,\cdot),(\mathbb{Q}^+,\max,\cdot),(\mathbb{R}^+,\max,\cdot)$ we mean that $\mathbb{Z}^+,\mathbb{Q}^+, \mathbb{R}^+$ have the usual multiplication and $x+y = \max\{x,y\}$ (maximum of x,y)

<u>Definition 1.1</u>. A triple $(S,+,\cdot)$ is said to be a <u>semiring</u> if and only if S is a set and $+$ (addition) and $\cdot$ **(multiplication) are** binary operations on S such that,

(a) $(S,+)$ and $(S,\cdot)$ are commutative semigroups,

(b) for all $x,y,z \in S$, $(x+y)\cdot z = x\cdot z + y\cdot z$.

<u>Definition 1.2</u>. A semiring $(D,+,\cdot)$ is said to be a <u>ratio semiring</u> if and only if $(D,\cdot)$ is a group.

Definition 1.3. Let S be a semiring. Then $x \in S$ is said to be additively cancellative (A.C.) if and only if for all $y, z \in S$ ($x+y = x+z$ implies $y = z$). S is said to be additively cancellative (A.C.) if and only if every $x \in S$ is A.C..

Definition 1.4. Let S be a semiring. Then $x \in S$ is said to be multiplicatively cancellative (M.C.) if and only if for all $y, z \in S$ ($xy = xz$ implies $y = z$). S is said to be multiplicatively cancellative (M.C.) if and only if every $x \in S$ is M.C..

Definition 1.5. Let S be a semiring with a multiplicative zero 0. Then S is said to be zero multiplicatively cancellative (0-M.C.) if and only if for all $x, y, z \in S$, $xy = xz$ and $x \neq 0$ imply $y = z$.

Definition 1.6. Let S be a semiring. Then S is said to be strongly multiplicatively cancellative (S.M.C.) if and only if for all $x_1, x_2, y_1, y_2 \in S$, $x_1y_1 + x_2y_2 = x_1y_2 + x_2y_1$ implies $x_1 = x_2$ or $y_1 = y_2$.

Proposition 1.7. Let S be a semiring. If S is S.M.C. then S is M.C. or 0-M.C..

Proof. Suppose that S has a mutiplicative zero 0. We must show that S is 0-M.C. Let $x_1, y_1, y_2 \in S$ be such that $x_1y_1 = x_1y_2$ and $x_1 \neq 0$. Then $x_1y_1 + 0y_2 = x_1y_2 + 0y_1$. Since S is S.M.C., $x_1 = 0$ or $y_1 = y_2$. But $x_1 \neq 0$, so $y_1 = y_2$. Hence S is 0-M.C. Assume that S has no multiplicative zero. We must show that S is M.C. Let $x_1, y_1, y_2 \in S$ be such that $x_1y_1 = x_1y_2$. We must show that $y_1 = y_2$. Since S has no multiplicative zero then there is a $u \in S$ such that $ux_1 \neq x_1$. We have that $ux_1y_1 = ux_1y_2$. Let $x_2 = ux_1$. Then $x_1 \neq x_2$ and $x_2y_1 = x_2y_2$. Therefore $x_1y_1+x_2y_2 = x_1y_2+x_2y_1$. Since S is S.M.C., $x_1 = x_2$ or $y_1 = y_2$. But $x_1 \neq x_2$, so $y_1 = y_2$. Hence S is M.C.. #

Definition 1.8. Let S be a semiring with multiplicative identity 1. Then S is said to be precise if and only if for all $u, v \in S$, $1+uv = u+v$ implies $u = 1$ or $v = 1$.

Theorem 1.9. There is no finite ratio semiring of order greater than 1.

See [1], pages 5-11.

Corollary 1.10. If S is a finite semiring of order greater than 1, then S can not be M.C.

See [1], page 11.

The following theorem is well-known in semigroup theorem.

Theorem 1.11. Every finite cancellative semigroup is a group.

Theorem 1.12. If S is a semiring then S can be embedded into a ratio semiring if and only if S is M.C..

See [1], pages 12-14..

Assume that S is M.C.. Define a binary relation $\sim$ on $S \times S$ by $(x,y) \sim (x',y')$ if and only if $xy' = x'y$ for all $x,y,x',y' \in S$. In Theorem 1.12 it is shown that $\sim$ is an equivalence relation. Let $\alpha, \beta \in \frac{S \times S}{\sim}$. Define + and $\cdot$ on $\frac{S \times S}{\sim}$ in the following way: Choose $(a,b) \in \alpha$ and $(c,d) \in \beta$. Define $\alpha+\beta = [(ad+bc, bd)]$ and $\alpha\beta = [(ac, bd)]$. Theorem 1.12 has shown that $(\frac{S \times S}{\sim}, +, \cdot)$ is a ratio semiring.

Definition 1.13. The ratio semiring $\frac{S \times S}{\sim}$ in Theorem 1.12 is called the quotient ratio semiring of S and is denoted by QR(S).

Theorem 1.12 gives a natural embedding $f: S \longrightarrow QR(S)$ as foolows: Fix $c \in S$. If $x \in S$, define $f(x) = [(xc, c)]$.
Theorem 1.12 shows that f is independent of the choice of $c \in S$.
We identify S with f(S) so we can consider $S \subseteq QR(S)$.
Theorem 1.12 also gives the following remarks:

1) $[(c,c)]$ is the multiplicative identity of QR(S).

2) $[(a,b)]$ is the multiplicative inverse of $[(b,a)]$.

Therefore $[(c,xc)]$ is the multiplicative inverse of $[(xc,c)]$.

3) Let $\alpha \in QR(S)$ and choose $(x,y) \in \alpha$.

Then $\alpha = [(x,y)] = [(cx,c)] \cdot [(c,cy)] = f(x)f(y)^{-1}$.
Since we identify $x \in S$ with $f(x) \in QR(S)$ we can write $\alpha = xy^{-1}$.
This is well-defined, because if $(x',y') \in \alpha$ then $xy' = x'y$.
Hence we shall use the notation $[(x,y)] = \frac{x}{y}$ where $\frac{x}{y}$ means xy^{-1}.

Theorem 1.14. If S is an M.C. semiring then QR(S) is the smallest ratio semiring containing S up to isomorphism.

See [1], pages 14-15.

Proposition 1.15. Let S be an M.C. semiring. Then the following hold:

(a) If S is A.C., then QR(S) is A.C..

(b) S is S.M.C. if and only if QR(S) is precise.

Proof. (a) See [1], page 43.

(b) Assume that S is S.M.C.. We must show that QR(S) is precise. Let $\alpha, \beta \in QR(S)$ be such that $1+\alpha\beta = \alpha+\beta$ and choose $(a,b) \in \alpha$, $(c,d) \in \beta$. Then $[(bd+ac,bd)] = [(ad+bd,bd)]$,

$(bd+ac)bd = (ad+bc)bd$. Hence $bd+ac = bc+ad$. Since S is S.M.C., we get that $a = b$ or $c = d$. Thus $\alpha = 1$ or $\beta = 1$.

Conversely, assume that QR(S) is precise. Let $x_1, x_2, y_1, y_2 \in S$ be such that $x_1y_1+x_2y_2 = x_1y_2+x_2y_1$. Let $f: S \longrightarrow QR(S)$ be the natural embedding, so $f(x_1y_1+x_2y_2) = f(x_1y_2+x_2y_1)$. Hence $1+f(x_1)^{-1}f(y_1)^{-1}f(x_2)f(y_2) = f(y_1)^{-1}f(y_2)+f(x_1)^{-1}f(x_2)$. Since QR(S) is precise, we get that $f(x_1)^{-1}f(x_2) = 1$ or $f(y_1)^{-1}f(y_2) = 1$. Therefore $x_1 = x_2$ or $y_1 = y_2$. Hence S is S.M.C.. #

Theorem 1.16. If S is a semiring then S can be embedded into a ring if and only if S is A.C..

See [1], pages 37-39.

Assume that S is A.C.. Define a binary relation $\sim$ on $S \times S$ by $(x,y) \sim (x',y')$ if and only if $x+y' = x'+y$ for all $x, x', y, y' \in S$. In Theorem 1.16 it is shown that $\sim$ is an equivalence relation. Let $\alpha, \beta \in \frac{S \times S}{\sim}$. Define + and $\cdot$ on $\frac{S \times S}{\sim}$ in the following way: Choose $(a,b) \in \alpha$ and $(c,d) \in \beta$. Define $\alpha+\beta = [(a+c, b+d)]$ and $\alpha\beta = [(ac+bd, ad+bc)]$. In Theorem 1.16 it has been shown that $(\frac{S \times S}{\sim}, +, \cdot)$ is a ring.

Definition 1.17. The ring $\frac{S \times S}{\sim}$ in Theorem 1.16 is called the difference ring of S and is denoted by D(S).

Theorem 1.16 gives a natural embedding $f: S \longrightarrow D(S)$ as follows: Fix $c \in S$. If $x \in S$ define $f(x) = [(x+c, c)]$. Theorem 1.16 shows that f is independent of the choice of $c \in S$. We identify S with f(S) so we can consider $S \subseteq D(S)$. Theorem 1.16 also gives the following remarks:

1) $[(c,c)]$ is the additive identity of $D(S)$.

2) $[(a,b)]$ is the additive inverse of $[(b,a)]$.

Therefore $[(c,x+c)]$ is the additive inverse of $[(x+c,c)]$.

3) Let $\alpha \in D(S)$. Choose $(x,y) \in \alpha$.

Then $\alpha = [(x,y)] = [(c+x,c)] + [(c,y+c)] = f(x) - f(y)$. Since we identify $x \in S$ with $f(x) \in D(S)$, we can write $\alpha = x-y$. This is well-defined because if $(x_1,y_1) \in \alpha$ then $x+y_1 = x_1+y$.

Hence we shall use the notation $[(x,y)] = x-y$.

<u>Theorem 1.18</u>. If S is an A.C. semiring then D(S) is the smallest ring containing S up to isomorphism.

See [1], pages 39-40.

<u>Proposition 1.19</u>. Let S be an A.C. semiring. If S has a multiplicative identity 1 then $[(1+1,1)]$ which we identify with 1 is the multiplicative identity of D(S).

<u>Proof</u>. Let $\alpha \in D(S)$. Choose $(a,b) \in \alpha$.

Then $[(1+1,1)][(a,b)] = [(a+a+b,b+b+a)] = [(a,b)]$.

<u>Definition 1.20</u>. Let S be a semiring and $d \in S$. Then $x \in S$ is said to be an <u>additive identity of d in S</u> if and only if $x+d = d$. The set of all additive identities of d in S is denoted by $I_S(d)$.

<u>Proposition 1.21</u>. Let S be a semiring and $d \in S$. Then $I_S(d) = \emptyset$ or $I_S(d)$ is additive subsemigroup of S.

See [2], page 7.

<u>Definition 1.22</u>. A semiring $(K,+,\cdot)$ is said to be a <u>semifield</u> if and only if there exists an element a in K such thar $(K-\{a\},\cdot)$ is a group. If we wish to specify the element $a \in K$ such that

$(K-\{a\},\cdot)$ is a group we shall say that K is a semifield with respect to a.

<u>Theorem 1.23</u>. Let $(K,+,\cdot)$ be a semifield with respect to a. Then exactly one of the following holds:

1) $ax = a$ for all $x \in K$, or

2) $ax = x$ for all $x \in K$, or

3) $a^2 \neq a$ and $ae \neq a$ where e is the identity of $(K-\{a\},\cdot)$.

See [2], pages 10-11.

From Theorem 1.23 we see that there are three types of semifields with respect to a:

(1) Semifields with $ax = a$ for all $x \in K$ (<u>called type I semifields w.r.t. a</u>).

(2) Semifields with $ax = x$ for all $x \in K$ (<u>called type II semifields w.r.t. a</u>).

(3) Semifields with $a^2 \neq a$ and $ae \neq a$ where e is the identity of $(K-\{a\},\cdot)$ (<u>called type III semifields w.r.t. a</u>).

<u>Theorem 1.24</u>. Let K be a type I semifield w.r.t. a. Then a is an additive zero or a is an additive identity.

See [1], page 21.

<u>Definition 1.25</u>. Let S be a semiring. Then $a \in S$ is called an <u>infinity element</u> if and only if a is a multiplicative zero and an additive zero.

<u>Definition 1.26</u>. Let S be a semiring. Then $a \in S$ is called a <u>zero element</u> if and only if a is a multiplicative zero and an additive identity.

From Theorem 1.24, if a is an infinity element then we call K an <u>infinity-semifield</u> (∞-semifield) and if a is a zero element then we call K a <u>zero-semifield</u> (0-semifield).

<u>Theorem 1.27</u>. Let S be a semiring with a multiplicative zero a. Then S can be embedded into a type I semifield if and only if S is 0-M.C..

See [1], pages 27-28.

Assume that S is a 0-M.C. and $|S|>1$. Define a binary relation $\sim$ on $S\times(S-\{a\})$ by $(x,y)\sim(x',y')$ if and only if $xy' = x'y$ for all $x,x',y,y'\in S$. In Theorem 1.27 we obtain that $\sim$ is an equivalence relation.

Let $\alpha,\beta\in \frac{S\times(S-\{a\})}{\sim}$. Define + and $\cdot$ on $\frac{S\times(S-\{a\})}{\sim}$ in the following way: Choose $(a,b)\in\alpha$, $(c,d)\in\beta$. Define $\alpha+\beta = [(ad+bc,bd)]$ and $\alpha\beta = [(ac,bd)]$. In Theorem 1.27 it has been shown that $(\frac{S\times(S-\{a\})}{\sim},+,\cdot)$ is a type I semifield.

<u>Definition 1.28</u>. In Theorem 1.27, if a is the zero element then $\frac{S\times(S-\{a\})}{\sim}$ is called the <u>quotient 0-semifield of S</u> and is denoted by Q(S) and if a is the infinity element then $\frac{S\times(S-\{a\})}{\sim}$ is called the <u>quotient ∞-semifield of S</u> and is denoted by $Q_\infty(S)$.

The same remarks concerning QR(S) discussed after Definition 1.13 are also true for Q(S) and $Q_\infty(S)$. It is also true that $[(x,y)]$ is the zero element of Q(S) if and only if $x = a$, and $[(x,y)]$ is the infinity element of $Q_\infty(S)$ if and only if $x = a$.

Theorem 1.29. If S is O-M.C. semiring with a multiplicative zero a then $\frac{S\times(S-\{a\})}{\sim}$ is the smallest type I semifield containing S up to isomorphism.

See [1], pages 28-29.

Theorem 1.30. Let K be a O-semifield. Then K can be embedded into a field if and only if K is A.C. and precise.

See [1], page 43.

Proposition 1.31. Let S be a O-M.C. semiring. Then the following hold:

a) If S is A.C. then Q(S) is A.C..

b) S is S.M.C. if and only if Q(S) is precise.

The proof of this proposition is similar to the proof of Proposition 1.15.

Theorem 1.32. Let $(K,+,\cdot)$ be a semifield of type I or type II w.r.t. a of order > 2. If there is an element b in K such that $(K-\{b\},\cdot)$ is a group then $b = a$.

See [2], page 13.

Theorem 1.33. Let $(K,+,\cdot)$ be a semifield of type III w.r.t. a. If there exists an element b in K such that $(K-\{b\},\cdot)$ is a group then $b = a$.

See [2], page 13.

Proposition 1.34. Let K be a semifield of type II w.r.t. a and let $x,y \in K$. Then $xy = a$ if and only if $x = a$ and $y = a$.

See [2], page 19.

Proposition 1.35. Let K be a semifield with respect to a. Then K is a semifield of type III w.r.t. a if and only if there exists a unique d in $K-\{a\}$ such that $ax = dx$ for all x in K.

See [2], page 12.

Proposition 1.36. Let K be a type III semifield w.r.t. a. Then $xy \neq a$ for all $x, y \in K$.

See [2], page 19.

Theorem 1.37. Let K be a semifield of type II w.r.t. a and let e be the identity of $(K-\{a\}, \cdot)$. Then the following hold:

(1) If $a+a = a$, then $(K,+)$ is a band.

(2) If $a+a \neq a$, then $a+a = e+e$, and for all $x, y \in K-\{a\}$, $x+x = y+y$ if and only if $x = y$.

(3) $a+x = a$ or $a+x = e+x$ for all $x \neq a$.

See [2], page 20.

Theorem 1.38. Let K be a semifield of type II w.r.t. a and let e be the identity of $(K-\{a\}, \cdot)$. Define $D = K-\{a\}$ and $S = \{x \in D \mid a+x = a\}$. Then the following hold:

(1) $S = \emptyset$ or S is an additive subsemigroup of $I_D(e)$.

(2) If $e \in S$ then $S = I_D(e)$.

(3) $D-S = \emptyset$ or $D-S$ is an ideal of $(D,+)$.

See [2], pages 20-21.

Theorem 1.39. Let D be a ratio semiring, a a symbol not representing any element in D and let $S \subseteq I_D(1)$ have the property that either $S = \emptyset$ or S is an additive subsemigroup of $I_D(1)$ such that $D-S$ is an ideal of $(D,+)$ if D is infinite. Then we can extend the binary operations of D to $K = D \cup \{a\}$ making K into a semifield of type II w.r.t. a such that

(1) $ax = xa = x$ for all $x \in K$,

(2) $a+x = x+a = a$ for all $x \in S$ and

$a+x = x+a = 1+x$ for all $x \in D-S$.,

(3) $a+a = \begin{cases} a \text{ or } 1 & \text{if } 1+1 = 1, \\ 1+1 & \text{if } 1+1 \neq 1. \end{cases}$

See [2], pages 23-29.

Theorem 1.40. Let K be a semifield of type III w.r.t. a and let $d \in K-\{a\}$ be such that $ax = dx$ for all $x \in K$. Then the following hold:

(1) If $a+a = a$, then $(K,+)$ is a band.

(2) If $a+a \neq a$, then $a+a = d+d$, and for all $x,y \in K-\{a\}$, $x+x = y+y$ if and only if $x = y$.

(3) $a+x = a$ or $a+x = d+x$ for all $x \neq a$.

See [2], pages 30-31.

Theorem 1.41. Let K be a semifield of type III w.r.t. a and let $d \in K-\{a\}$ be such that $ax = dx$ for all $x \in K$. Let $D = K-\{a\}$ and $S = \{x \in D \mid a+x = a\}$. Then the following hold:

(1) $S = \emptyset$ or S is an additive subsemigroup of $I_D(d)$.

(2) If $d \in S$, then $S = I_D(d)$.

(3) $D-S = \emptyset$ or $D-S$ is an ideal of $(D,+)$.

See [2], pages 31-32.

Theorem 1.42. Let D be a ratio semiring, a a symbol not representing any element of D, $d \in D$ and let $S \subseteq I_D(d)$ have the property that either $S = \emptyset$ or S is an additive subsemigroup of $I_D(d)$ such that $D-S$ is an ideal of $(D,+)$ if D is infinite. Then we can extend the binary operations of D to $K = D \cup \{a\}$ making K into a semifield of type III w.r.t. a such that

(1) $ax = xa = dx$ for all $x \in D$ and $a^2 = d^2$,

(2) $a+x = x+a = a$ for all $x \in S$ and

$a+x = x+a = d+x$ for all $x \in D-S$,

(3) $a+a = \begin{cases} a \text{ or } d & \text{if } 1+1 = 1, \\ d+d & \text{if } 1+1 \neq 1. \end{cases}$

See [2], pages 34-41.

Theorem 1.43. Every ratio semiring can be embedded into a 0-semifield.

See [2], pages 41-46.

Theorem 1.44. Every ratio semiring can be embedded into an ∞-semifield.

See [2], pages 46-47.

Corollary 1.45. If S is an M.C. semiring, then S can be embedded into a type I semifield.

Proof. Since S is M.C., S can be embedded into a quotient ratio semiring QR(S). By Theorem 1.43 and Theorem 1.44, QR(S) can be embedded into a type I semifield and hence so can S.

Proposition 1.46. If S is an M.C. semiring, then S can be embedded into a type II semifield.

See [2], page 47.

Proposition 1.47. If S is an M.C. semiring, then S can be embedded into a type III semifield.

See [2], pages 48.

Assume S is an M.C. semiring. Then QR(S) exists. Let a be a symbol not representing any element of QR(S). Extend + and $\cdot$ from QR(S) to K = QR(S) $\cup$ {a} as follows:

(1) ax = xa = a for all $x \in K$, and x+a = a+x = x for all $x \in K$. Then K is a 0-semifield. (by Theorem 1.43).

(2) ax = xa = a for all $x \in K$, and x+a = a+x = a for all $x \in K$. Then K is an ∞-semifield. (by Theorem 1.44).

(3) ax = xa = x for all $x \in K$, and x+a = a+x = 1+x for all $x \in K$ (where 1 is the multiplicative identity of QR(S)). Then K is a type II semifield. (by Proposition 1.46).

(4) Fix $d \in QR(S)$. Define ax = xa = dx for all $x \in K$ and x+a = a+x = d+x for all $x \in K$. Then K is a type III semifield (by Proposition 1.47).

Let $f: S \longrightarrow QR(S)$ be the natural embedding. Since $QR(S) \subseteq K$, we can consider $f: S \longrightarrow K$. Hence S can be embedded into any type of semifield.