CHAPTER V

A GENERALIZATION OF A

STRUCTURE THEOREM OF RISCH

5.1 Preliminary Lemmas

In this chapter we generalize a structure theorem of Risch to another class of fields. First we give some preliminary lemmas that are used in the proof of the main result.

Lemma 5.1.1. Let F be a differential field of characteristic zero and K a differential extension field of F. Let $t \in K$ be transcendental over F. Assume that there are $u_1,\dots,u_p$, $v_1,\dots,v_p$ in K, with $u_1,\dots,u_p$ nonzero, such that precisely one member in each pair (u_i,v_i) is algebraic over $F(t, u_1,\dots,u_{i-1}, v_1,\dots,v_{i-1})$ if $i > 1$ or algebraic over $F(t)$ if $i = 1$. Then the elements $du_1/u_1 - dv_1,\dots,du_p/u_p - dv_p$, dt of $\Omega_{K/F}$ are linearly independent over K

Proof. Let $a_0, a_1,\dots,a_p \in K$ be such that

$$(5.1) \qquad a_0dt + a_1(du_1/u_1 - dv_1) + \cdots + a_p(du_p/u_p - dv_p) = 0.$$

Since, by Theorem 1.6, there exists a K - homomorphism $g : \Omega_{K/F} \to \Omega_{K/F(t,u_1,\dots,u_{p-1},v_1,\dots,v_{p-1})}$ such that $g \circ d = d_p$, where d_p is the $F(t, u_1,\dots,u_{p-1}, v_1,\dots,v_{p-1})$- derivation of K into $\Omega_{K/F(t,u_1,\dots,u_{p-1},v_1,\dots,v_{p-1})}$.

Compose (5.1) by g to get

$$a_p(d_pu_p/u_p - d_pv_p) = 0.$$

Since precisely one member of each pair (u_p, v_p) is algebraic over

$F(t, u_1,...,u_{p-1},v_1,...,v_{p-1})$, by Theorem 1.7, we obtain $d_pu_p/u_p-d_pv_p \neq 0$. Hence $a_p = 0$. Proceeding in the same manner, we conclude that $a_i = 0$ for all $i = 0, 1, ...,p$. #

Lemma 5.1.2. Let F be a differential field of characteristic zero and K a differential extension field of F. Assume that there are $u_1,...,u_p$, $v_1,...,v_p$ in K, with $u_1,...,u_p$ nonzero, such that precisely one member in each pair (u_i, v_i) is algebraic over $F(u_1,..., u_{i-1}, v_1,...,v_{i-1})$ if $i > 1$ or algebraic over F if $i = 1$. Then the elements $du_1/u_1 - dv_1,...,du_p/u_p - dv_p$ of $\Omega_{K/F}$ are linearly independent over K.

Proof. The proof is analogous to the proof of Lemma 5.1.1.

Definition 5.1.3. Let F be a differential field and K a differential extension field of F. An element t of K is called an <u>elementary integral</u> over F if there exist elements v_0, $v_1,...,v_n$ in F, with $v_1,..,v_n$ nonzero, and $c_1,...,c_n$ constants of F such that

$$D(t) = D(v_0) + \sum_{i=1}^{n} c_iD(v_i)/v_i \qquad \text{for each given derivation D of K.}$$

We say that K is a <u>general elementary extension</u> of F if there exists a finite tower of fields $F = F_0 \subset F_1 \subset \cdots \subset F_n = K$ such that for each i, $1 \leq i \leq n$, $F_i = F_{i-1}(t_i)$ and one of the following holds:

(i) t_i is algebraic over F_{i-1},

(ii) $t_i = \exp(u)$ for some u in F_{i-1},

(iii) $t_i = \log(u)$ for some nonzero u in F_{i-1},

(iv) t_i is primitive and nonelementary integral over F_{i-1}.

Let F be a differential field. The statement "$K = F(t_1,...,t_n)$ is an elementary extension of F (or general elementary extension of F)" refers to the tower of fields $F = F_0 \subset F_1 \subset \cdots \subset F_n = K$ such that $F_i = F_{i-1}(t_i)$ where each t_i satisfies one of the conditions in the definition of the elementary extension (or general elementary extension, respectively).

Let $E = \{ i \mid t_i = \exp(a_i), a_i \in F_{i-1}, 1 \le i \le n \}$ and

$L = \{ i \mid t_i = \log(a_i), a_i \in F_{i-1}, 1 \le i \le n \}$.

Example. Let **C** be the field of complex numbers and let $K = \mathbf{C}(x, \log(x), \exp(x), \sqrt{x})$ with the usual derivation $D = d/dx$.

Note that $\mathbf{C} = F_0 \subset F_1 = F_0(t_1) \subset F_2 = F_1(t_2) \subset F_3 = F_2(t_3) \subset F_4 = F_3(t_4) = K$, where $t_1 = x$, $t_2 = \log(x)$, $t_3 = \exp(x)$, $t_4 = \sqrt{x}$. Hence $E = \{3\}$ and $L = \{2\}$.

Lemma 5.1.4. Let F be a differential field of characteristic zero, $K = F(t_1,\ldots,t_n)$ an elementary extension of F with the same subfield of constants C. Suppose that $u_1,\ldots,u_m$ and v are elements of K, with $u_1,\ldots, u_m$ nonzero, and $\alpha_1 = 1, \alpha_2,\ldots,\alpha_m \in C$ such that $\sum_{i=1}^{m} \alpha_i D(u_i)/u_i + D(v) \in F$ for each given derivation D of K. Then

(1) there are $\{ c_i \in C \mid i \in L \}$, $\{ d_i \in C \mid i \in E \}$ and f in the algebraic closure of F in K such that

$$v + \sum_{i \in L} c_i t_i + \sum_{i \in E} d_i a_i = f,$$

where $t_i = \exp(a_i)$ for $i \in E$,

(2) there are $\{ k_i \in \mathbf{Z} \mid k_1 \neq 0, i = 1,\ldots,m \}$, $\{n_i \in \mathbf{Z} \mid i \in L \}$, $\{ m_i \in \mathbf{Z} \mid i \in E \}$ and g in the algebraic closure of F in K such that

$$\left(\prod_{i=1}^{m} u_i^{k_i}\right)\left(\prod_{i \in L} a_i^{n_i}\right)\left(\prod_{i \in E} t_i^{m_i}\right) = g,$$

where $t_i = \exp(a_i)$ for $i \in L$.

Proof. There are $z_1,\ldots,z_r, y_1,\ldots,y_r \in K$ with $z_i \neq 0$ for $i = 1,\ldots,r$ such that

(i) $D(z_i)/z_i = D(y_i)$ for each given derivation D of K,

(ii) one member in each pair (z_i, y_i) must be t_j for some $1 \leq j \leq n$,

(iii) precisely one member in each pair (z_i, y_i) is algebraic over

$F(z_1, \ldots, z_{i-1}, y_1, \ldots, y_{i-1})$ if $i > 1$ or algebraic over F if $i = 1$,

(iv) K is algebraic over $F(z_1, \ldots, z_r, y_1, \ldots, y_r)$.

We have $D(z_i)/z_i - D(y_i) = 0 \in F$ for $i = 1, \ldots, r$, and $\sum_{i=1}^{m} \alpha_i D(u_i)/u_i + D(v) \in F$.

Since $u_1, \ldots, u_m, v \in K$ and by (iv), $u_1, \ldots, u_m, v$ are algebraic over $F(z_1, \ldots, z_r, y_1, \ldots, y_r)$. Hence tr.deg. $F(z_1, \ldots, z_r, y_1, \ldots, y_r, u_1, \ldots, u_m, v)/F < r+1$.

By Theorem 1.8, we can conclude that the elements $dz_1/z_1 - dy_1, \ldots, dz_r/z_r - dy_r$, $\sum_{i=1}^{m} \alpha_i du_i/u_i + dv$ of $\Omega_{K/F}$ are linearly dependent over C.

Thus there exist $\gamma_0, \gamma_1, \ldots, \gamma_r \in C$, not all zero, such that

$$\gamma_0\left(\sum_{i=1}^{m} \alpha_i du_i/u_i + dv\right) + \sum_{i=1}^{r} \gamma_i(dz_i/z_i - dy_i) = 0. \tag{5.2}$$

Claim that $\gamma_0 \neq 0$. Suppose not. Then

$$\sum_{i=1}^{r} \gamma_i(dz_i/z_i - dy_i) = 0 \tag{5.3}$$

By Lemma 5.1.2, $dz_1/z_1 - dy_1, \ldots, dz_r/z_r - dy_r$ are linearly independent over K.

Hence, from (5.3), $\gamma_i = 0$ for $i = 1, \ldots, r$.

Thus $\gamma_0 = \gamma_1 = \ldots = \gamma_r = 0$, a contradiction. So we have the claim.

Without loss of generality, we assume $\gamma_0 = 1$. Let $c_1 = 1, c_2, \ldots, c_s$ be a basis for the vector space over $\mathbf{Q}$ spanned by $\{\alpha_1 = 1, \alpha_2, \ldots, \alpha_m, \gamma_1, \ldots, \gamma_r\}$ and

for $i = 1, \ldots, m$, write $\alpha_i = \sum_{j=1}^{s} n_{ij}c_j$ with each $n_{ij} \in \mathbf{Q}$, and

for $i = 1, \ldots, r$, write $\gamma_i = \sum_{j=1}^{s} m_{ij}c_j$ with each $m_{ij} \in \mathbf{Q}$.

Replacing each c_j by c_j/least common denominator of $\{n_{11}, \ldots, n_{ms}, m_{11}, \ldots, m_{rs}\}$, if necessary, we can assume each n_{ij} , $m_{ij} \in \mathbf{Z}$. In particular,

$$1 = \alpha_1 = \sum_{j=1}^{s} n_{1j}c_j = n_{11}c_1 \ ; \text{ that is } n_{11} = 1, n_{12} = n_{13} = \ldots = n_{1s} = 0.$$

$$\text{Let} \quad w_j = \left(\prod_{i=1}^{m} u_i^{n_{ij}}\right)\left(\prod_{i=1}^{r} z_i^{m_{ij}}\right) \quad \text{for } j = 1,\ldots,s.$$

We can rewrite (5.2) as $\sum_{j=1}^{s} c_j\, dw_j/w_j + d(v - \gamma_1 y_1 - \cdots - \gamma_r y_r) = 0.$

By Theorem 1.7, $v - \gamma_1 y_1 - \cdots - \gamma_r y_r$ and w_1 are algebraic over F.

Hence

$$(5.4) \quad \begin{cases} v - \gamma_1 y_1 - \cdots - \gamma_r y_r = f \quad \text{and} \\ \left(\prod_{i=1}^{m} u_i^{n_{i1}}\right)\left(\prod_{i=1}^{r} z_i^{m_{i1}}\right) = w_1 = g, \end{cases}$$

where f, g are in the algebraic closure of F in K. Note that each y_i either equals some t_j with $j \in L$ or equals some a_j where $t_j = \exp(a_j)$, $j \in E$. Each z_i either equals some t_j with $j \in E$ or equals some a_j where $t_j = \log(a_j)$, $j \in L$. Substitute the y_i and the z_i in (5.4), to obtain the desired result. #

5.2 Main Theorem

Theorem. Let F be a differential field of characteristic zero, $K = F(t_1,\ldots,t_n)$ a general elementary extension of F with the same subfield of constants C. Suppose that $u_1,\ldots,u_m$ and v are elements of K, and $\alpha_1 = 1, \alpha_2, \ldots, \alpha_m \in C$ are such that

$$\sum_{i=1}^{m} \alpha_i D(u_i)/u_i + D(v) = 0 \quad \text{for each given derivation D of K.}$$

Then

(1) there are $\{ c_i \in C \mid i \in L\}$, $\{ d_i \in C \mid i \in E \}$ and f in the algebraic closure of

F in K such that

$$v + \sum_{i\in L} c_i t_i + \sum_{i\in E} d_i a_i = f,$$

where $t_i = \exp(a_i)$ for $i \in E$,

(2) there are $\{ k_i \in \mathbf{Z} \mid k_1 \neq 0, i = 1,\ldots,m \}$, $\{ n_i \in \mathbf{Z} \mid i \in L \}$, $\{m_i \in \mathbf{Z} \mid i \in E\}$, and g in the algebraic closure of F in K such that

$$\left(\prod_{i=1}^{m} u_i^{k_i}\right)\left(\prod_{i\in L} a_i^{n_i}\right)\left(\prod_{i\in E} t_i^{m_i}\right) = g,$$

where $t_i = \exp(a_i)$ for $i \in L$.

Proof. The proof is by induction on r, the number of primitive and nonelementary integral among $t_1,\ldots,t_n$. For $r = 0$, the theorem is true by Lemma 5.1.4.

Let $r \geq 1$. Assume that the theorem is true for all smaller values of r. Among the r t_i's that are primitive and nonelementary integral over F_{i-1}, let t_R be the one with the largest subscript.

For notation simplicity, let $F' = F(t_1,\ldots,t_{R-1})$ if $R > 1$ or $F' = F$ if $R = 1$ and $t = t_R$.

There are $z_1,\ldots,z_p, y_1,\ldots, y_p \in K$ with $z_i \neq 0$ for $i=1,\ldots,p$ such that

(i) for $i = 1,\ldots,p$, $D(z_i)/z_i = D(y_i)$ for each given derivation D of K,

(ii) one member in each pair (z_i, y_i) must be t_j for some $R+1 \leq j \leq n$,

(iii) precisely one member in each pair (z_i, y_i) is algebraic over $F'(t, z_1,\ldots,z_{i-1}, y_1,\ldots,y_{i-1})$ if $i > 1$ or algebraic over $F'(t)$ if $i = 1$,

(iv) K is algebraic over $F'(t, z_1,\ldots,z_p, y_1,\ldots,y_p)$.

Note that $D(t) \in F'$,

$$D(z_i)/z_i - D(y_i) = 0 \in F' \text{ for } i = 1,\ldots,p,$$

$$\sum_{i=1}^{m} \alpha_i D(u_i)/u_i - D(v) = 0 \in F',$$

and that tr.deg. $F'(t, z_1,\dots,z_p, y_1,\dots,y_p,u_1,\dots,u_m, v)/F' < p + 2$. By Theorem 1.8, we can conclude that the elements $dt, dz_1/z_1 - dy_1,\dots,dz_p/z_p - dy_p, \sum_{i=1}^{m}\alpha_i du_i/u_i - dv$ of $\Omega_{K/F'}$ are linearly dependent over C, and so

there exist $\gamma_o, \gamma_1, \dots,\gamma_p, \gamma \in C$, not all zero, such that

$$(5.5)\qquad \gamma_o\Big(\sum_{i=1}^{m}\alpha_i du_i/u_i - dv\Big) + \sum_{i=1}^{p}\gamma_i(dz_i/z_i - dy_i) + \gamma dt = 0.$$

Claim that $\gamma_o \neq 0$. Suppose not. Then

$$(5.6)\qquad \sum_{i=1}^{p}\gamma_i(dz_i/z_i - dy_i) + \gamma dt = 0.$$

By Lemma 5.1.1, $dz_1/z_1 - dy_1,\dots,dz_p/z_p - dy_p, dt$ are linearly independent over K.
Hence, from (5.6), $\gamma_1 = \gamma_2 = \dots = \gamma_p = \gamma = 0$.
Thus $\gamma_o = \gamma_1 = \gamma_2 = \dots = \gamma_p = \gamma = 0$, a contradiction. So we have the claim.
Without loss of generality, we assume $\gamma_o = 1$.
Let $c_1 = 1, c_2,\dots,c_q$ be a basis for the vector space over **Q** spanned by $\{\alpha_1 = 1,\alpha_2,\dots,\alpha_m,\gamma_1,\dots,\gamma_p\}$ and for each $i = 1,\dots,m$, write $\alpha_i = \sum_{j=1}^{q} n_{ij}c_j$ with each $n_{ij} \in \mathbf{Q}$, for each $i = 1,\dots,p$, write $\gamma_i = \sum_{j=1}^{q} m_{ij}c_j$ with each $m_{ij} \in \mathbf{Q}$.

Replacing each c_j by c_j/least common denominator of $\{n_{11},\dots,n_{mq},m_{11},\dots,m_{pq}\}$, if necessary, we can assume each $n_{ij}, m_{ij} \in \mathbf{Z}$. In particular,

$$1 = \alpha_1 = \sum_{j=1}^{q} n_{1j}c_j = n_{11}c_1;\ \text{that is } n_{11} = 1,\ n_{12} = \dots = n_{1q} = 0.$$

Let
$$w_0 = v - \sum_{i=1}^{p}\gamma_i y_i + \gamma t,$$

$$w_j = \left(\prod_{i=1}^{m} u_i^{n_{ij}}\right)\left(\prod_{i=1}^{p} z_i^{m_{ij}}\right) \qquad \text{for } j = 1,\dots,p.$$

We can rewrite (5.5) as

$$(5.7) \qquad \sum_{j=1}^{q} c_j dw_j/w_j \; + \; dw_0 \; = \; 0.$$

By Theorem 1.7, $w_0, w_1, \ldots, w_q$ are algebraic over F'.

We now show $\gamma = 0$.

Note that $\gamma D(t) = \sum_{j=1}^{q} c_j D(w_j)/w_j + D(w_0)$ for each given derivation D of K.

$w_0, w_1, \ldots, w_q$ are algebraic over F', so taking σ, an element of the Galois group of $F'(w_0, w_1, \ldots, w_q)$ over F', and summing over all σ, gives

$$s\gamma\, Dt \; = \; \sum_{j=1}^{q} c_j\, D(Nw_j)/Nw_j \; + \; D(Tw_0),$$

for some $s \in \mathbf{Z}\backslash\{0\}$ and for each given derivation D of K (N and T denote the norm and trace respectively).

If $\gamma \neq 0$, then t would be an elementary integral over F', a contradiction. Hence $\gamma = 0$.

Consequently, $\sum_{j=1}^{q} c_j D(w_j)/w_j + D(w_0) = 0$ for each given derivation D of K and $w_0 = v - \sum_{i=1}^{p} \gamma_i y_i$.

The number of primitive and nonelementary integral among $t_1, \ldots, t_{R-1}, w_0, \ldots, w_q$, the generators of $F(t_1, \ldots, t_{R-1}, w_0, \ldots, w_q)$ over F, is less than r.

So by the induction hypothesis we have,

$$(5.8) \qquad w_0 + \sum_{i \in L'} c_i' t_i \; + \; \sum_{i \in E'} d_i' a_i \; = \; f,$$

$$(5.9) \qquad \left(\prod_{i=1}^{q} w_i^{k_i}\right)\left(\prod_{i \in L'} a_i^{n_i}\right)\left(\prod_{i \in E'} t_i^{m_i}\right) \; = \; g,$$

where $E' = \{\, i \mid 1 \leq i \leq R-1,\ t_i = \exp(a_i) \text{ and } a_i \in F_{i-1} \,\}$,

$L' = \{ i \mid 1 \le i \le R\text{-}1,\ t_i = \log(a_i) \text{ and } a_i \in F_{i-1} \}$,

$c_i', d_i' \in C$, $n_i, m_i, k_i \in \mathbf{Z}$ with $k_1 \neq 0$ and f, g are in the algebraic closure of F in K.

Now recall

(5.10) $$w_0 = v - \sum_{i=1}^{p} \gamma_i y_i$$

(5.11)
$$\begin{cases} w_1 = u_1^{n_{11}} u_2^{n_{21}} \cdots u_m^{n_{m1}} \prod_{i=1}^{p} z_i^{m_{i1}} & (n_{11} = 1), \\ w_2 = u_1^{n_{12}} u_2^{n_{22}} \cdots u_m^{n_{m2}} \prod_{i=1}^{p} z_i^{m_{i2}} & (n_{12} = 0), \\ \vdots & \\ w_q = u_1^{n_{1q}} u_2^{n_{2q}} \cdots u_m^{n_{mq}} \prod_{i=1}^{p} z_i^{m_{iq}} & (n_{1q} = 0). \end{cases}$$

Subsitute the expressions (5.10) and (5.11) in (5.8) and (5.9) respectively and note that each y_i either equals some t_j with $j \in L$ or equals some a_j where $t_j = \exp(a_j)$, $j \in E$, and each z_i either equals some t_j with $j \in E$ or equals some a_j where $t_j = \log(a_j)$, $j \in L$,we get the desired result.

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