APPENDIX A

CONSTANT FIELD

There are three major extensions in tower of fields analyzed in this thesis, the first is the adjoining an algebraic element, the second is the adjunction of a primitive element, and the last is the adjunction of an exponential element. The following lemmas give relations among the subfield of constants between an extension field and the ground field.

Lemma A-1. Let F be a differential field and K a differential extension field of F. Suppose that K is algebraic over F. Then the subfield of constants of K is algebraic over the subfield of constants of F.

Proof. Let $c \neq 0$ be a constant of K. There exist $a_0, a_1, \ldots, a_{n-1}$ in F such that $X^n + a_{n-1}X^{n-1} + \cdots + a_0$ is the minimal polynomial of c.

Hence $c^n + a_{n-1}c^{n-1} + \cdots + a_0 = 0$.

Differentiation gives $D(a_{n-1})c^{n-1} + \cdots + D(a_0) = 0$ for each given derivation D of K. By the minimality of n, we have for i = 0, 1, ...,n-1, $D(a_i) = 0$ for each given derivation D. Thus for i = 0, 1, ...,n-1, a_i is a constant of F. Therefore c is algebraic over the subfield of constants of F. #

Lemma A-2. Let F be a differential field of characteristic zero. Let t be an element in a differential extension field of F such that $D(t) \in F$ for each given derivation D. Assume that there is no b in F such that $D(t) = D(b)$ for each given derivation D. Then t is transcendental over F and F(t) has the same subfield of constants as F.

Proof. Suppose that t is algebraic over F. There exist $a_0, a_1, \ldots, a_{n-1}$ in F such that $X^n + a_{n-1}X^{n-1} + \cdots + a_0$ is the minimal polynomial of t.

Hence $t^n + a_{n-1}t^{n-1} + \cdots + a_0 = 0$. For each given derivation D, we get

$$(nD(t) + D(a_{n-1}))t^{n-1} + ((n-1)a_{n-1}D(t) + D(a_{n-2}))t^{n-2} + \cdots + D(a_0) = 0.$$

So $D(t) = D\left(\frac{-a_{n-1}}{n}\right)$ for each given derivation D. Since $\frac{-a_{n-1}}{n}$ is in F, this contradicts the hypothesis of the lemma, hence t is transcendental over F.

Next, we will show that F(t) has the same subfield of constants as F.

Suppose that there exists a constant c in F(t) that is not in F.

Case 1: $c = b_m t^m + b_{m-1}t^{m-1} + \cdots + b_0$ with $b_0, b_1, \ldots, b_m$ in F, integer $m > 0$ and $b_m \neq 0$.

For each given derivation D,

$$0 = D(c) = D(b_m)t^m + (mb_mD(t) + D(b_{m-1}))t^{m-1} + \cdots + D(b_0).$$

So $D(b_m) = mb_mD(t) + D(b_{m-1}) = 0$ for each given derivation D.

Therefore $D(t) = D\left(\frac{-b_{m-1}}{mb_m}\right)$ for each given derivation D. Since $\frac{-b_{m-1}}{mb_m}$ is in F, this again contradicts the hypothesis of the lemma.

Case 2: $c = f/g$ with f, g relatively prime elements of F[t], g not in F and g monic.

For each given derivation D, we have $D(f/g) = 0$. So $gD(f) = fD(g)$. Clearly, D(f) and D(g) are also in F[t] and have degrees respectively at most the degree of f and less than the degree of g. Relative primeness implies $g|D(g)$, so that $D(g) = 0$ for each given derivation D, and proceed as in case 1 to get a contradiction.

#

Lemma A-3. Let F be a differential field of characteristic zero. Let t be a nonzero element in a differential extension field of F such that $D(t)/t = \alpha_D$ for some α_D in F. Assume that there does not exist nonzero element y in F satisfying $D(y)/y = k\alpha_D$ for each given derivation D, and for all positive integer k. Then t is transcendental over F and F(t) has the same subfield of constants as F.

Proof. Suppose that t is algebraic over F. There exist $b_0, b_1, \ldots, b_{n-1}$ in F such that $X^n + b_{n-1}X^{n-1} + \cdots + b_0$ is the minimal polynomial of t.

Hence $t^n + b_{n-1}t^{n-1} + \cdots + b_0 = 0$. For each given derivation D,

$$n\alpha_D t^n + \sum_{k=1}^{n-1} (D(b_k) + k\, b_k \alpha_D)t^k + D(b_0) = 0.$$

Note that $\alpha_D \neq 0$ for each given derivation D. Thus we get that t satisfy

$$X^n + \sum_{k=1}^{n-1} \frac{(D(b_k) + k\, b_k \alpha_D)}{n\alpha_D} X^k + \frac{D(b_0)}{n\alpha_D} = 0.$$

By the uniqueness of the minimal polynomial of t, we have $\dfrac{D(b_0)}{n\alpha_D} = b_0$ for each given derivation D. So $D(b_0)/b_0 = n\alpha_D$ for each given derivation D, which is a contradiction. Therefore t is transcendental over F.

Next, we will show that F(t) has the same subfield of constants as F. Suppose that there exists a constant c in F(t) that is not in F. Write $c = f/g$ with f, g relatively prime elements of F[t]. For each given derivation D, we have $D(f/g) = 0$, so that $gD(f) = fD(g)$. Note that for each given derivation D, D(f) and D(g) are elements in F[t] of degrees at most those of f, and g respectively. Relative primeness implies $g|D(g)$, and $f|D(f)$ for each given derivation D.

Claim that f is a monomial. Suppose not. Let $a_n t^n$ and $a_m t^m$ be two distinct terms in f where a_n, a_m nonzero elements in F and n, m are integers such that $0 \leq n < m$. Since for each given derivation D, $f|D(f)$, we get

$$D(a_n)/a_n + nD(t)/t \; = \; D(a_m)/a_m + mD(t)/t.$$

So $(m-n)\, D(t)/t \; = \; D(a_n a_m^{-1})/(a_n a_m^{-1})$ for each given derivation D, a contradiction.

So f is a monomial. Similarly, g is a monomial.

Now, write $c \; = \; at^n$ where a is a nonzero element in F and n is a nonzero integer. For each given derivation D,

$$0 \; = \; D(c) \; = \; [D(a) + na\alpha_D]t^n.$$

Hence $D(a)/a \; = \; -n\alpha_D$ for each given derivation D, a contradiction.

Therefore F(t) has the same subfield of constants as F. #

APPENDIX B

In the following, we present a method to examine whether a certain function is nonelementary. The main tool is based on the following lemma, whose proof we refer to [1].

Lemma B-1 ([1]). Let F be a differential field of characteristic zero with a derivation D. Let F(t) be a differential extension field of F with the same subfield of constants and with t transcendental over F and either $D(t) \in F$ or $D(t)/t \in F$. Let $c_1,\dots,c_n \in F$ be linearly independent over $\mathbf{Q}$ and let $u_1,\dots,u_n$ be nonzero elements of F(t), $v \in F(t)$. Suppose that $\sum_{i=1}^{n} c_i D(u_i)/u_i + D(v) \in F[t]$. Then

(1) $v \in F[t]$,

(2) if $D(t) \in F$, then $u_i \in F$ for $i = 1,\dots, n$,

if $D(t)/t \in F$, then for each $i = 1,\dots,n$ there exists $m_i \in \mathbf{Z}$ such that $u_i/t^{m_i} \in F$.

Theorem B-2. Let F be a differential field of characteristic zero, with a derivation D. Let F(t) be a differential extension field of F with the same subfield of constants C, with t transcendental over F and $D(t) \in F$. Let $f \in F$. If $\int ft$ is elementary over F(t), then there exist $c \in C$ and $b \in F$ such that $f = D(ct + b)$.

Proof. Since $\int ft$ is elementary over F(t), by Theorem 2.1.1,there exist constants $c_1,\dots,c_n$ in F and elements $u_1,\dots,u_n$, v in F(t) , with $u_1,\dots,u_n$ nonzero, such that

$$ft = \sum_{i=1}^{n} c_i D(u_i)/u_i + D(v).$$

Without loss of generality, we may assume that the c_i are linearly independent over **Q**. By Lemma B-1, $v \in F[t]$ and $u_i \in F$ for $i = 1, \ldots, n$.

So $\sum_{i=1}^{n} c_i D(u_i)/u_i \in F$. Hence

(*) $\qquad ft = D(v) + (\text{elements of } F).$

Write $v = \sum_{j=0}^{m} b_j t^j$ where $b_j \in F$ for $j = 0, \ldots, m$ and $b_m \neq 0$. Thus

$$D(v) = D(b_m)t^m + (mb_m D(t) + D(b_{m-1}))t^{m-1} + (\text{elements of } F[t] \text{ of degree} < m\text{-}1).$$

Claim that $m \leq 2$. Suppose that $m > 2$.

Then $D(b_m) = 0$ and $mb_m D(t) + D(b_{m-1}) = 0$. Hence b_m is a constant, and $D(mb_m t + b_{m-1}) = 0$. Implying $mb_m t + b_{m-1}$ is a constant, and hence t is algebraic over F, a contradiction. So we have the claim.

Thus $v = b_2 t^2 + b_1 t + b_o$ and

$$D(v) = D(b_2)t^2 + (2b_2 D(t) + D(b_1))t + b_1 D(t) + D(b_o).$$

Clearly b_2 is a constant. Substituting $D(v)$ in (*) and equate the term of t yield $f = D(2b_2 t + b_1)$. #

Example. Let **C** be the field of complex numbers and let $F = \mathbf{C}(x)$ with the usual derivation $D = d/dx$. Let $t = \log(x)$. Then by Lemma A-2, t is transcendental over F and F(t) has the same subfield of constants as F.

We will show that $\int \frac{\log(x)}{x+1}$ is nonelementary over F(t).

Suppose that $\int \frac{\log(x)}{x+1}$ is elementary. Then there exist $c \in \mathbf{C}$ and $b \in F$ such that

$$(*) \qquad \frac{1}{x+1} = \frac{c}{x} + D(b).$$

Assume first that x+1 occurs in the denominator of b, then it occurs at least twice in the denominator of D(b) which is not balanced by $\frac{1}{x+1}$. If x+1 does not occur in the denominator of b, then it is clear that x+1 does not occur in the denominator of D(b). Both cases are is impossible. Thus, in either case the equation (*) has no solutions.

Hence $\int \frac{\log(x)}{x+1}$ is nonelementary over F(t).

APPENDIX C

1. Elementary Extension

A differential field K is an elementary extension of a differential field F if there exists a finite tower of fields $F = F_0 \subset F_1 \subset \cdots \subset F_n = K$ such that for i,with $1 \leq i \leq n$, $F_i = F_{i-1}(t_i)$ and one of the following holds:

(i) t_i is algebraic over F_{i-1},

(ii) $t_i = \exp(u)$ for some u in F_{i-1},

(iii) $t_i = \log(u)$ for some nonzero u in F_{i-1}.

2. More Generalized Extensions

Let F be a differential field with derivation D and the subfield of constants C. Let A and B be finite indexing sets. Let K be a differential extension of F.

2.1 $\mathcal{EL}$ - Elementary Extension

Let $\mathcal{E} = \{G_\alpha(\exp R_\alpha(Y)) \mid \alpha \in A\}$,

$\mathcal{L} = \{H_\beta(\log S_\beta(Y)) \mid \beta \in B\}$

be sets of expressions where :

(1) $G_\alpha, R_\alpha, H_\beta, S_\beta$ are in C(Y) for all $\alpha \in A, \beta \in B$,

(2) for all $\beta \in B$, if $H_\beta(Y) = P_\beta(Y)/Q_\beta(Y)$ with P_β, Q_β in C[Y] and $Q_\beta \neq 0$, then $\deg P_\beta \leq \deg Q_\beta + 1$.

We call K an $\mathcal{E}\mathcal{L}$ - elementary extension of F if there exists a finite tower of fields $F = F_0 \subset F_1 \subset \cdots \subset F_n = K$ such that for i, with $1 \leq i \leq n$, $F_i = F_{i-1}(t_i)$ and one of the following holds:

(i) t_i is algebraic over F_{i-1},

(ii) $t_i = \exp(u)$ for some u in F_{i-1},

(iii) $t_i = \log(u)$ for some nonzero u in F_{i-1},

(iv) for some $\alpha \in A$, there are u and nonzero v in F_{i-1} such that

$D(t_i) = D(u)G_\alpha(v)$ where $v = \exp R_\alpha(u)$,

(for brevity, $t_i = \int G_\alpha(\exp R_\alpha(u))D(u)$),

(v) for some $\beta \in B$, there are u, v in F_{i-1} such that $D(t_i) = D(u)H_\beta(v)$

where $v = \log S_\beta(u)$ and $S_\beta(u) \neq 0$,

(for brevity, $t_i = \int H_\beta(\log S_\beta(u)D(u))$.

2.2 Ei - Extension

Let $\mathcal{E} = \{G_\alpha(\exp R_\alpha(Y)) \mid \alpha \in A\}$,

$\mathcal{L} = \{H_\beta(\log S_\beta(Y)) \mid \beta \in B\}$

be sets of expressions where :

(1) $G_\alpha, R_\alpha, H_\beta, S_\beta$ are in C(Y) for all $\alpha \in A, \beta \in B$,

(2) for all $\beta \in B$, if $H_\beta(Y) = P_\beta(Y)/Q_\beta(Y)$ with P_β, Q_β in C[Y] and $Q_\beta \neq 0$, then deg $P_\beta \leq$ deg Q_β.

We call K an <u>Ei - extension</u> of F if there exists a finite tower of fields $F = F_o \subset F_1 \subset \cdots \subset F_n = K$ such that for i, with $1 \leq i \leq n$, $F_i = F_{i-1}(t_i)$ and one of the following holds:

(i) t_i is algebraic over F_{i-1},

(ii) $t_i = \exp(u)$ for some u in F_{i-1},

(iii) $t_i = \log(u)$ for some nonzero u in F_{i-1},

(iv) for some $\alpha \in A$, there are u and nonzero v in F_{i-1} such that

$D(t_i) = D(u)G_\alpha(v)$ where $v = \exp R_\alpha(u)$,

(for brevity, $t_i = \int G_\alpha(\exp R_\alpha(u))D(u)$),

(v) for some $\beta \in B$, there are u, v in F_{i-1} such that $D(t_i) = D(u)H_\beta(v)$

where $v = \log S_\beta(u)$ and $S_\beta(u) \neq 0$,

(for brevity, $t_i = \int H_\beta(\log S_\beta(u))D(u)$),

(vi) for some $\alpha \in A$, there are nonzero u, v in F_{i-1} such that

$D(t_i) = (D(u)/u)G_\alpha(v)$ where $v = \exp R_\alpha(u)$,

(for brevity, $t_i = \int G_\alpha(\exp R_\alpha(u))D(u)/u$),

(vii) for some $\beta \in B$, there are nonzero u, v in F_{i-1} such that

$D(t_i) = (D(u)/u)H_\beta(v)$ where $v = \log S_\beta(u)$ and $S_\beta(u) \neq 0$,

(for brevity, $t_i = \int H_\beta(\log S_\beta(u))D(u)/u$).

2.3 Gamma Extension

We say that K is a Gamma extension of F if there exists a finite tower of fields $F = F_0 \subset F_1 \subset \cdots \subset F_n = K$ such that for i, with $1 \le i \le n$, $F_i = F_{i-1}(t_i)$ and one of the following holds:

(i) t_i is algebraic over F_{i-1},

(ii) $t_i = \exp(u)$ for some u in F_{i-1},

(iii) $t_i = \log(u)$ for some nonzero u in F_{i-1},

(iv) there are $G \in C(Y)$, u and nonzero v in F_{i-1} and $r \in \mathbf{Q}$ with $-1 \le r \le 1$ such that $D(t_i) = D(u^r)G(v)$ where $v = \exp(u)$, (for brevity, $t_i = \int D(u^r)G(\exp(u))$).

2.4 Generalized Log - Explicit Extension

An element x in K is called a primitive over F if $D(x) \in F$. Let $t \in K$ be primitive over F. We call t a simple logarithm over F if there exist $u_1,\ldots,u_m$ in F such that for some constant c in K, $t + c \in \bar{F}(\log(u_1),\ldots,\log(u_m))$. We say that t is nonsimple if it is not a simple logarithm over F.

We call K a generalized log - explicit extension of F if there exists a finite tower of fields $F = F_0 \subset F_1 \subset \cdots \subset F_n = K$ such that for i, with $1 \le i \le n$, $F_i = F_{i-1}(t_i)$ and one of the following holds:

(i) t_i is algebraic over F_{i-1},

(ii) $t_i = \exp(u)$ for some u in F_{i-1},

(iii) $t_i = \log(u)$ for some nonzero u in F_{i-1},

(iv) t_i is primitive and nonsimple over F_{i-1}.

2.5 General Elementary Extension

Let t be an element in a differential extension field of F. We call t an elementary integral over F if there exist elements $v_0, v_1, \ldots, v_n$ in F, with $v_1, \ldots, v_n$ nonzero, and $c_1, \ldots, c_n$ constants of F such that $D(t) = D(v_0) + \sum_{i=1}^{n} c_i D(v_i)/v_i$.

We say that t is nonelementary integral over F if it is not elementary integral over F.

We call K a general elementary extension of F if there exists a finite tower of fields $F = F_0 \subset F_1 \subset \cdots \subset F_n = K$ such that for i, with $1 \leq i \leq n$, $F_i = F_{i-1}(t_i)$ and one of the following holds:

(i) t_i is algebraic over F_{i-1},

(ii) $t_i = \exp(u)$ for some u in F_{i-1},

(iii) $t_i = \log(u)$ for some nonzero u in F_{i-1},

(iv) t_i is primitive and nonelementary integral over F_{i-1}.

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