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BOUNDARIES OF OVERLAPPING REULEAUX TRIANGLES

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In this paper, we investigate an analogous case of a problem proposed by J. W. Fickett in 1980, i.e. finding an interval of the ratio

$$\frac{\text{length}\,(\partial R_1 \cap \text{Int}\,(R_2))}{\text{length}\,(\partial R_2 \cap \text{Int}\,(R_1))},$$

where R_1 and R_2 are two congruent Reuleaux triangle such that $\text{Int}\,(R_1) \cap \text{Int}\,(R_2) \neq \varnothing$. Denote ∂R_i and $\text{Int}\,(R_i)$ the boundary and the interior of R_i, respectively.

We finish the proof when R_2 is a translated copy R_1 and we obtain some interesting results when R_1 and R_2 intersect in general position.

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CHAPTER I

INTRODUCTION

In 1980, J. W. Fickett proposed the following problem in [3]:

Find an interval of the ratio

$$\frac{\text{length}\,(\partial R_1 \cap \text{Int}\,(R_2))}{\text{length}\,(\partial R_2 \cap \text{Int}\,(R_1))},$$

where R_1 *and* R_2 *are two congruent rectangular regions whose interior intersect. Denote* ∂R_i *and* $\text{Int}\,(R_i)$ *the boundary and the interior of region* R_i*, respectively.*

He also conjectured that all possible values of the above ratio must lies between $\frac{1}{3}$ and 3.

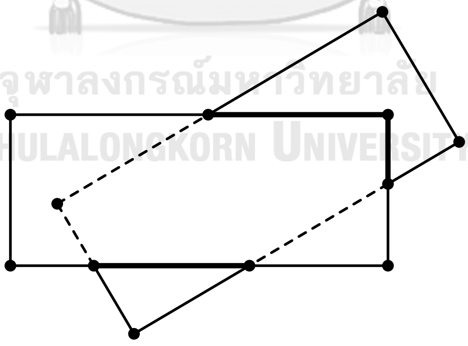


Figure 1.1: The main objective of the Fickett's problem is to find an interval of the ratio between the length of dashed segments and that of thick segments.

Then, in 2004, C. Nielsen and C. Powers studied the same problem in another case, i.e. in the case of R_1 and R_2 are two congruent equilateral triangles (as illustrated in figure 1.2). They have proved in [4] that

$$\frac{1}{2} \leq \frac{\text{length}\,(\partial R_1 \cap \text{Int}\,(R_2))}{\text{length}\,(\partial R_2 \cap \text{Int}\,(R_1))} \leq 2,$$

for any two congruent equilaterals R_1 and R_2 with nonempty intersection of their interiors.

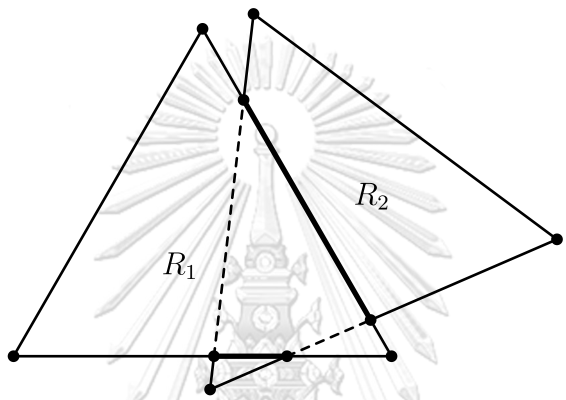


Figure 1.2: Two congruent equilaterals R_1 and R_2 whose interiors intersect are given. According to [4], the ratio between the length of dashed segments and the length of thick segments always lies between $\frac{1}{2}$ and 2.

In this paper, we are going to investigate the Fickett's problem in the case of R_1 and R_2 are two congruent Reuleaux triangles with nonempty intersection of their interiors by distinguishing the investigation into two parts :

1. when R_2 is an image of translation of R_1, and

2. when R_1 and R_2 intersect in general position.

CHAPTER II

REULEAUX TRIANGLE AND ITS PROPERTIES

In this chapter, we are going to introduce a construction of Reuleaux triangle and some of its properties.

2.1 A Construction of Reuleaux Triangle

A Reuleaux triangle is a convex region whose boundary consists of three vertices of an equilateral, any two of them are connected by a circular arc which is a part of a circle centered at the other vertex with radius equal to the side length of the equilateral.

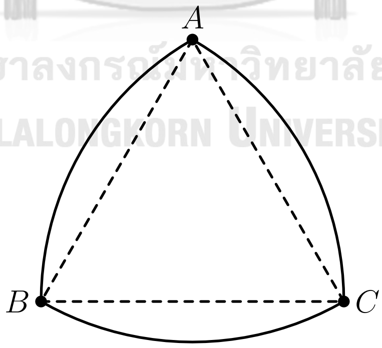


Figure 2.1: The boundary of Reuleaux triangle ABC is shown in the solid arcs. Note that A, B, C are three vertices of an equilateral $\triangle ABC$ (dashed), and any two of them arc connected by circular arc centered at the other vertex as shown.

Note 1. For convenience, in this paper, we denote Reu(ABC) the Reuleaux triangle whose vertices are A, B and C.

2.2 Some Properties of Reuleaux Triangle

Reuleaux triangle is a convex region which satisfies a property called **constant width**, i.e. the distance between two parallel supporting lines of the region is always constant.

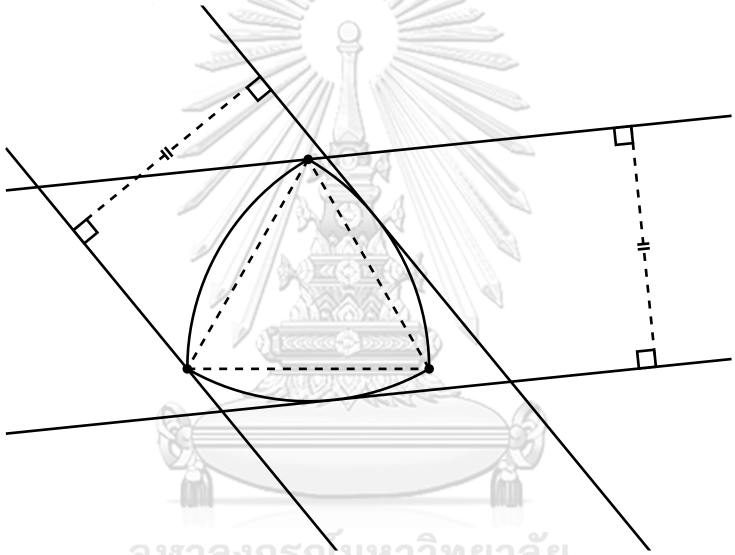


Figure 2.2: The distance between two parallel supporting lines of Reuleaux triangle is always constant.

Another elementary example of convex region with constant width is a circle since the distance between its two parallel supporting lines is equal to its diameter.

Note that, according to this property, the distance between two distinct points in Reuleaux triangle does not exceed the width of the Reuleaux.

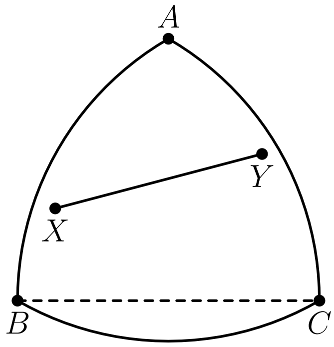
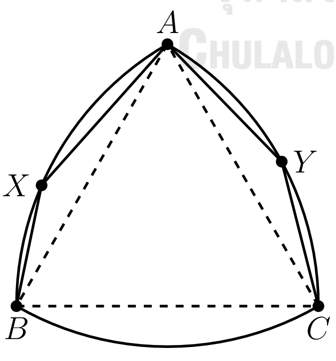


Figure 2.3: For any points X, Y in $\mathrm{Reu}\,(ABC)$, $\left|\overline{XY}\right| \leq \left|\overline{BC}\right|$. and the equality holds if and only if one of them is a vertex of $\mathrm{Reu}\,(ABC)$ and the other is a point on the opposite arc.

Moreover, we also obtain another basic property via the following propositions which can be easily proved by elementary geometry.

Proposition 2.1. *Let X and Y be two points on arc $\widehat{AB}$ and $\widehat{AC}$ of $\mathrm{Reu}\,(ABC)$, respectively. Then*



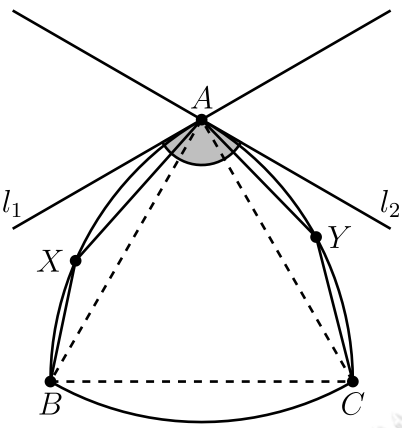
(i) $\dfrac{\pi}{3} \leq \angle XAY < \dfrac{2\pi}{3}$,

(ii) $\dfrac{\pi}{3} \leq \angle CBX < \dfrac{\pi}{2}$ *and, similarly,* $\dfrac{\pi}{3} \leq \angle BCY < \dfrac{\pi}{2}$,

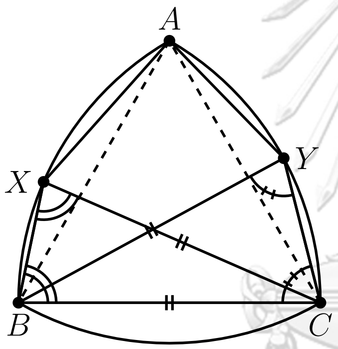
(iii) $\angle AXB = \angle AYC = \dfrac{5\pi}{6}$, *and*

(iv) the perimeter of $\mathrm{Reu}\,(ABC)$ *is equal to* $\pi\left|\overline{BC}\right|$.

Proof.



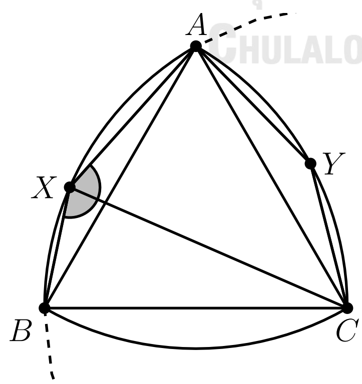
(i) Let l_1 and l_2 be the tangent lines at point A of arcs $\widehat{AB}$ and $\widehat{AC}$, respectively. Then $l_1 \perp AC$ and $l_2 \perp AB$. Since $\angle BAC = \frac{\pi}{3}$, the obtuse angle (shaded) between these two lines is equal to $\frac{2\pi}{3}$. Hence, $\frac{\pi}{3} = \angle BAC \leq \angle XAY < \frac{2\pi}{3}$ as desired.



(ii) Clearly, $\angle CBX > \angle CBA = \frac{\pi}{3}$. Since $|CB| = |CX|$, we have

$$\angle BXC = \angle XBC = \frac{\pi}{2} - \frac{1}{2}\angle BCX < \frac{\pi}{2}.$$

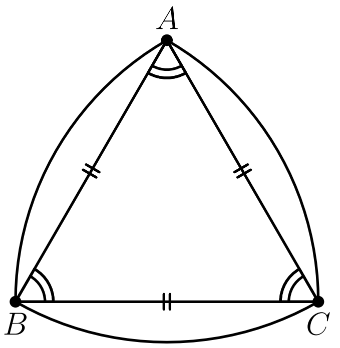
Similarly, $\frac{\pi}{3} < \angle BCY < \frac{\pi}{2}$.



(iii) Since C is the center of arc $\widehat{AB}$, we have $\angle XAB = \frac{1}{2}\angle BCX$ and $\angle XBA = \frac{1}{2}\angle ACX$. Hence,

$$\begin{aligned}
\angle AXB &= \pi - (\angle XAB + \angle XBA) \\
&= \pi - \frac{1}{2}(\angle BCX + \angle ACX) \\
&= \pi - \frac{1}{2}\left(\frac{\pi}{3}\right) = \frac{5\pi}{6}.
\end{aligned}$$

Similarly, $\angle AYC = \frac{5\pi}{6}$.



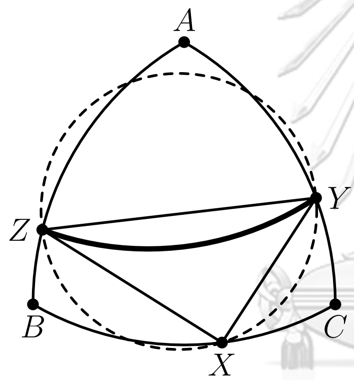
(iv) Since $\angle BCA = \frac{\pi}{3}$, we have $\left|\widehat{AB}\right| = \frac{\pi}{3}\left|\overline{BC}\right|$. Note that

$$\left|\widehat{AB}\right| = \left|\widehat{BC}\right| = \left|\widehat{CA}\right|.$$

Hence, the perimeter is equal to $3\left|\widehat{AB}\right| = \pi\left|\overline{BC}\right|$ as desired.

□

Proposition 2.2.



Let Reu (ABC) *be a Reuleaux of unit width and X, Y, Z three points on $\widehat{BC}$, $\widehat{CA}$ and $\widehat{AB}$, respectively. Then the circumradius of $\triangle XYZ$ does not exceed 1.*

Proof. Without loss of generality, assume $\overline{YZ}$ is the longest side of $\triangle XYZ$. Then $\angle ZXY$ is the largest angle of the triangle. Note that $\left|\overline{YZ}\right| \leq 1$.

Assume the contrary that the circumradius of $\triangle XYZ$ is greater than 1. Applying law of sine in $\triangle XYZ$, we obtain.

$$\frac{1}{\sin\angle ZXY} \geq \frac{YZ}{\sin\angle ZXY} > 2 = \frac{1}{\sin\angle BXC}$$

, since $\angle BXC = \frac{5\pi}{6}$ by proposition 2.1(iii). Hence, $\angle ZXY > \frac{5\pi}{6}$ which is a contradiction. □

We also obtain the following consequence from the above proposition.

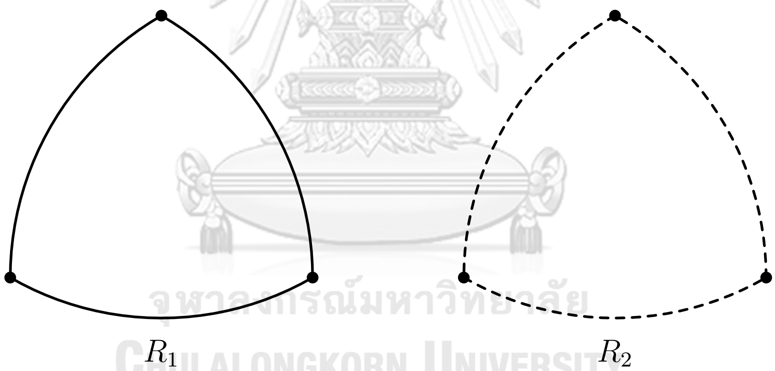
Corollary 2.3. *For any two points* $Y \neq B$ *and* $Z \neq C$ *on arcs* $\widehat{AC}$ *and* $\widehat{AB}$*, respectively, of* $\mathrm{Reu}\,(ABC)$ *of unit width, let* $\widehat{YZ}$ *be an arc of a unit circle which lies on opposite side of line* XY *to vertices* A *(shown as the thick arcs in proposition 2.2). Then* $\widehat{YZ}$ *never meets arc* $\widehat{BC}$ *of* $\mathrm{Reu}\,(ABC)$*. Moreover* $\left|\widehat{YZ}\right| < \left|\widehat{BC}\right|$ *and the center of* $\widehat{YZ}$ *lies outside* $\mathrm{Reu}\,(ABC)$*.*

CHAPTER III

INTERSECTIONS OF REULEAUX TRIANGLES

In this chapter, we are going to separate all cases of intersection between two congruent Reuleaux triangles by considering the numbers of arcs in the boundary of intersection area.

Clearly, all possible numbers of arcs on the boundary of the intersection region are at least 2 and at most 6.



Let R_1 and R_2 be two congruent Reuleaux triangles as shown above. For convenience, the boundaries of R_1 and R_2 will be illustrated as solid arcs and dashed arcs, respectively.

When $\text{Int}\,(R_1)$ and $\text{Int}\,(R_2)$ overlap together, the boundary of $\text{Int}\,(R_1) \cap \text{Int}\,(R_2)$

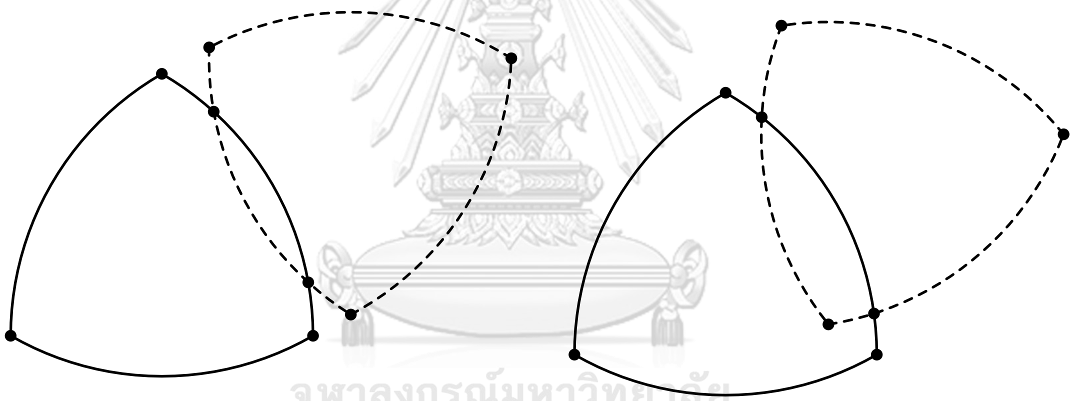
consists of solid arcs and dashed arcs. Denote

a = the number of solid arcs on the boundary of $\text{Int}(R_1) \cap \text{Int}(R_2)$, and

b = the number of dashed arcs on the boundary of $\text{Int}(R_1) \cap \text{Int}(R_2)$.

Without loss of generality, assume that $a \leq b$. Note that $1 \leq a, b \leq 3$. Next, we are going to distinguish all possible cases of ordered pair (a, b).

Case 1 : $a = 1$. The intersection which coresponding to $(1, 1)$ and $(1, 2)$ are illustrated in Figures 3.1a and 3.1b, respectively. Note that proposition 2.2 and corollary 2.3 guarantee that the case of intersection $(a, b) = (1, 2)$ exists.

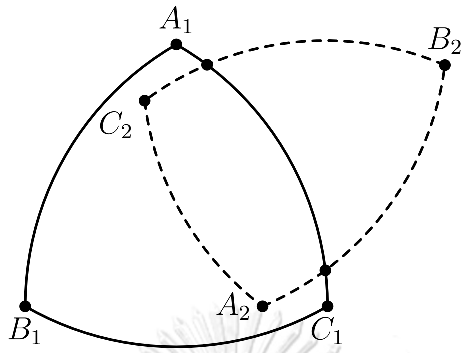


(a) $(a, b) = (1, 1)$

(b) $(a, b) = (1, 2)$

Figure 3.1

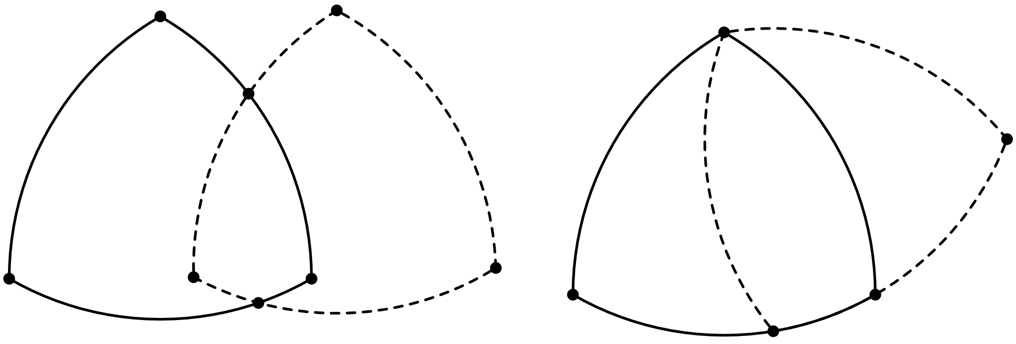
Note that if $b = 3$ then there are two vertices of R_2 which lie in R_1 as shown below.



This situation can occur only when these two vertices, say A_2 and C_2, of R_2 are also two vertices of R_1. If $\{A_2, C_2\} = \{A_1, C_1\}$, then the intersection will correspond to $(a, b) = (1, 1)$. Otherwise, R_1 and R_2 are coincide and the intersection area is corresponding to $(a, b) = (3, 3)$.

Hence there are no intersections corresponding to $(a, b) = (1, 3)$.

Case 2 : $a = 2$**.** An ordinary example of the intersection corresponding to $(a, b) = (2, 2)$ is illustrated in figure 3.2a.



(a) (b)

Figure 3.2: $(a,b) = (2,2)$

According to figure 3.2b, this kind of intersection can be seen that it corresponds to $(a,b) = (2,2)$ by looking at the lowest arc on the boundary of intersection consisting of two arcs, solid and dashed, overlapping each other.

Next, we are going to show that there are no intersection corresponding to $(a,b) = (2,3)$. Assume the contrary. Then the intersection can be illustrated as in the figure 3.3.

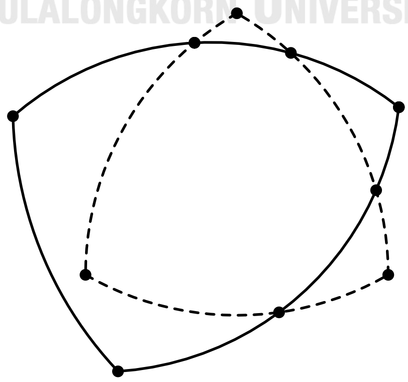


Figure 3.3

Note that there are a vertex of R_2 lying in R_1 and a part of its opposite arc lying in the interior of R_1, which contradicts the constant width property of Reuleaux triangle.

Case 3 : $a = 3$**.** Then $b = 3$ only, and an example of corresponding intersection is illustrated in figure 3.4.

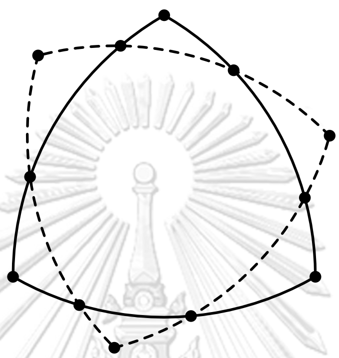


Figure 3.4: $(a, b) = (3, 3)$

Finally, we can distinguish all cases of intersection of two Reuleaux triangle as desired.

CHAPTER IV

FICKETT'S PROBLEM ON TRANSLATION OF REULEAUX TRIANGLES

In this section, we are going to investigate the Fickett's problem in the case of two congruent Reuleaux triangles each of which is an image via a transation of the other.

Note 2. In this section, without loss of generality, we assume that the width of the Reuleaux triangles is 1.

4.1 Distinguishing All Cases of Intersection

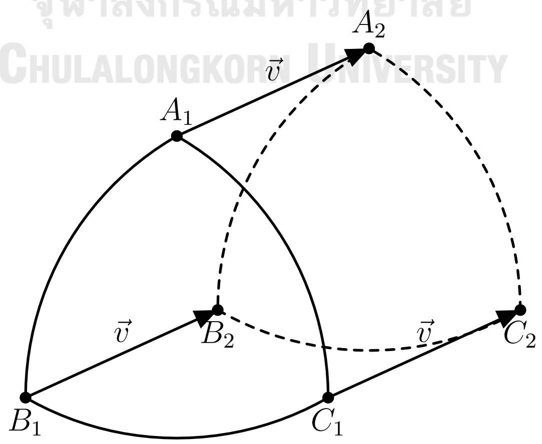


Figure 4.1

Let $R_1 = \text{Reu}(A_1B_1C_1)$ and $R_2 = \text{Reu}(A_2B_2C_2)$ be two congruent Reuleaux triangles such that R_2 is the image of translation of R_1 via vector $\vec{v} = \vec{A_1A_2} = \vec{B_1B_2} = \vec{C_1C_2}$ as shown in figure 4.1. Note that translation is a bijecctive map from R_1 to R_2 and preserves interior and boundary. Note that we consider the translation via nonzero vector only.

According to figure 4.1, we firstly begin with the following Lemma.

Lemma 4.1. *If* $|\vec{v}| \geq 1$, *then* $\text{Int}(R_1) \cap \text{Int}(R_2) = \varnothing$.

Proof. Assume the contrary. Let x be a point in $\text{Int}(R_1) \cap \text{Int}(R_2)$. Since $x \in \text{Int}(R_2)$, there exists $x' \in \text{Int}(R_1)$ such that $\vec{x'x} = \vec{v}$. Hence $\left|\vec{x'x}\right| = |\vec{v}| \geq 1$. Note that x' and x lie in the interior of R_1 whose width is 1, a contradiction. □

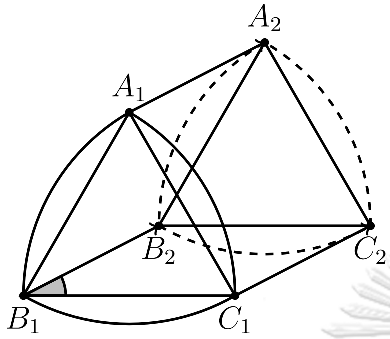
Remark 4.2. The result from lemma 4.1 is still true for general convex regions of unit width.

Now we obtain a consequence from the previous lemma that if the interior of two Reuleaux triangles intersect, then the magnitude of translation vector must less than 1. The next lemma helps us to distinguish all cases of intersection in this situation.

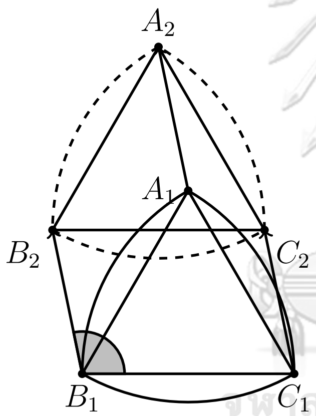
Lemma 4.3. *If* $\text{Int}(R_1) \cap \text{Int}(R_2) \neq \varnothing$, *then there is at least 1 vertex of a Reuleaux triangle on the boundary of intersection area.*

Proof. Clearly, there are no two vertices of the same Reuleaux triangle that lie simulateneously in the interior of the other Reuleuax triangle.

By symmetry, it suffices to assume that $0 \leq \angle C_1B_1B_2 < \pi$.

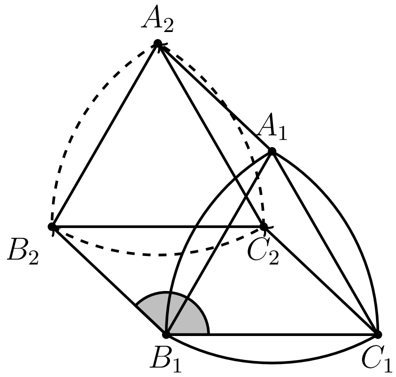


1. If $0 \leq \angle C_1B_1B_2 \leq \dfrac{\pi}{3}$, then B_2 lies in $\mathrm{Int}\,(R_1)$ since $\left|\overrightarrow{B_1B_2}\right| < 1$, and becomes a part of the boundary of intersection area as shown.



2. If $\dfrac{\pi}{3} < \angle C_1B_1B_2 \leq \dfrac{2\pi}{3}$, then $0 < \angle B_2A_2A_1 = \angle B_2B_1A_1 \leq \dfrac{\pi}{3}$.

   Using the same argument as 1., we obtain that A_1, a vertex of R_1, is a part of the boundary of intersection area.



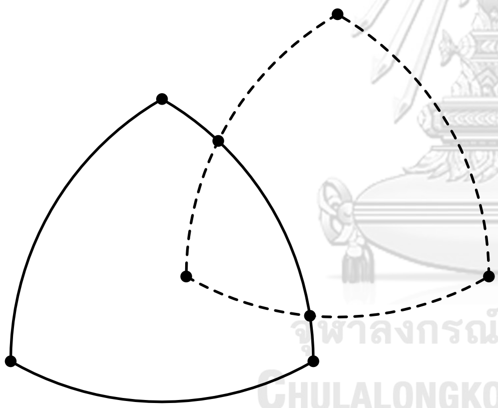
3. If $\dfrac{2\pi}{3} < \angle C_1B_1B_2 < \pi$, then $0 < \angle C_2C_1A_1 < \dfrac{\pi}{3}$.

   Using the same argument as 1., we obtain that C_2, a vertex of R_2, is a part of the boundary of intersection area.

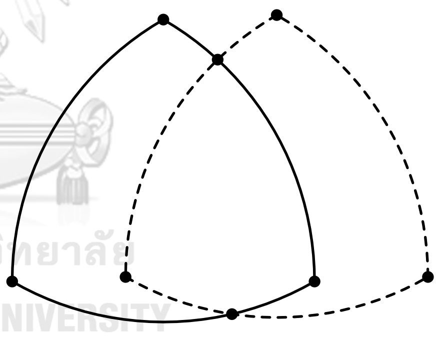
□

Note that the boundary of intersection area between two Reuleaux triangles which is an image of translation of each other must contain at least one vertex of a Reuleaux, so there are only 1 or 2 vertices of Reuleaux triangles on the boundary of overlapping region.

Hence, we obtain an important consequence from Lemma 4.3 that if each of the two Reuleaux triangles is an image of translation of one another, then the intersection between them must satisfy only one of the following two cases : $(a,b) = (1,2)$ or $(a,b) = (2,2)$.



(a) $(a,b) = (1,2)$



(b) $(a,b) = (2,2)$

Figure 4.2: If each of Reuleaux triangles is an image of translation of one another, there are only two cases of intersection occur.

4.2 Computing the Ratio

Now we look back to the figure in proposition 2.2 as shown below.

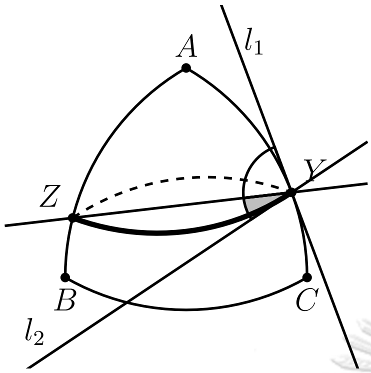


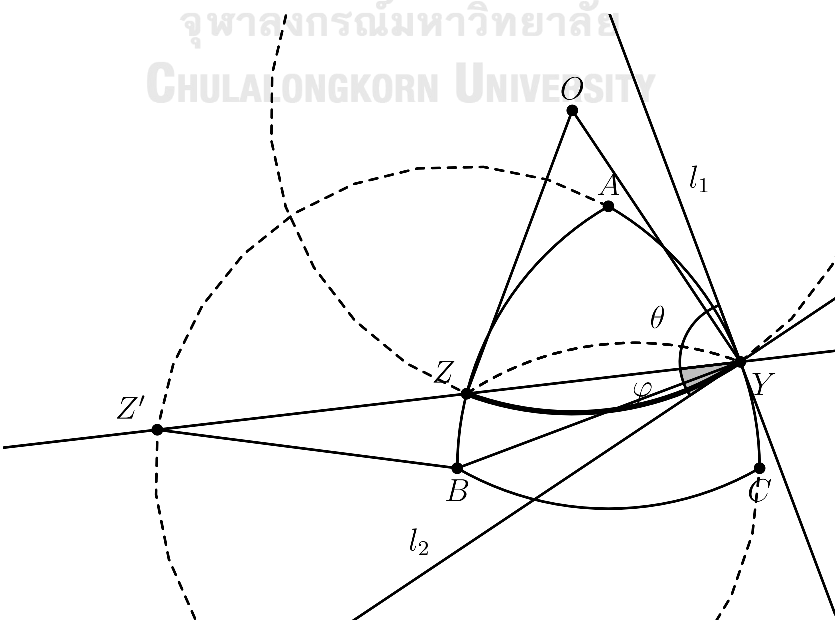
Figure 4.3

According to figure 4.3 on the left, we need to show that the image of refection of arc $\widehat{YZ}$ across line $\overleftrightarrow{ZY}$ lies in $\mathrm{Reu}\,(ABC)$.

Let l_1 be the tangent line of $\widehat{AC}$ at Y and l_2 the tangent line of $\widehat{YZ}$ at Y. It suffices to show that the white angle is greater than or equal to the gray angle.

Proposition 4.4. *The image of reflection of arc $\widehat{YZ}$ across line $\overleftrightarrow{ZY}$ lies in* $\mathrm{Reu}\,(ABC)$.

Proof.



Let θ and φ be the white angle and the gray angle, respectively. Let Γ_1 and Γ_2 be two unit circles which is the main circles of arcs $\widehat{AC}$ and $\widehat{YZ}$, respectively. Denote O the center of Γ_2 as shown.

Note that Z is a point on the interior of Γ_1. Hence, $\left|\overline{YZ}\right| < \left|\overline{YZ'}\right|$ where $Z' \neq Y$ is a second point of intersection between line $\overleftrightarrow{ZY}$ and Γ_1. Consequently, $2\theta = \angle YBZ > \angle YOZ = 2\varphi$ since l_1 is a tangent line of Γ_1 and l_2 is a tangent line of Γ_2, so we are done. $\square$

Reflecting circular sector YOZ across line $\overleftrightarrow{ZY}$, we obtain an illustration as shown in figure 4.4.

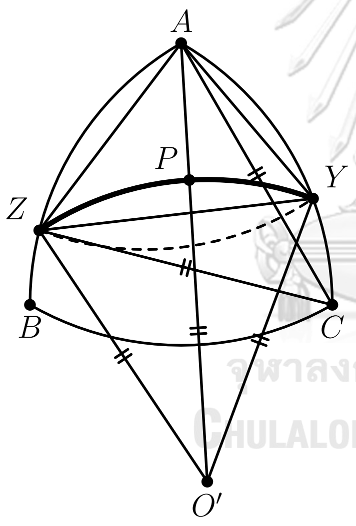


Figure 4.4

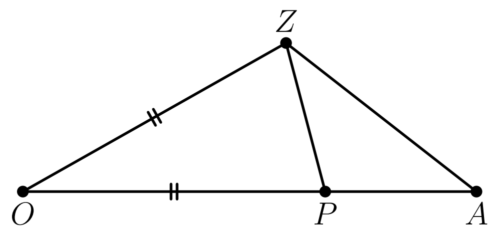
Let O' be the center of solid arc $\widehat{ZY}$ as shown on the left. Note that $\triangle ACZ$ and $\triangle YO'Z$ are isoscele triangle. Hence,

$$\begin{aligned} \angle AZO' &= \angle AZY + \angle YZO' \\ &\leq \angle AZC + \angle YZO' \\ &< \frac{\pi}{2} + \frac{\pi}{2} = \pi \end{aligned}$$

and, similarly, $\angle AYO' < \pi$. This implies $\overrightarrow{OA}$ always lies between $\overrightarrow{OY}$ and $\overrightarrow{OZ}$.

Thus, there is a point of intersection, namely P, between arc $\widehat{YZ}$ and segment $\overline{O'A}$ as illustrated in figure 4.4.

Proposition 4.5.



Let $\triangle ZOP$ be an isoscele triangle and A a point on the extension of $\overrightarrow{OP}$. Then $ZA \geq ZP$.

Proof. Note that $\angle ZPO$ is always acute and $\angle ZPO \geq \angle ZAP$. Applying law of sine in $\triangle XPA$, we obtain

$$\frac{ZA}{ZP} = \frac{\sin \angle ZPA}{\sin \angle ZAP} = \frac{\sin \angle ZPO}{\sin \angle ZAP} \geq 1$$

and the equality holds if and only if P coincides with A. $\square$

Corollary 4.6. *According to figure 4.4,* $\left|\widehat{ZY}\right| = \left|\widehat{ZP}\right| + \left|\widehat{PY}\right| \leq \left|\widehat{ZA}\right| + \left|\widehat{AY}\right|$. *The equality holds if and only if A coincides with Z or Y.*

By corollary 4.6, we now obtain a lower bound of the ratio between the length of boundaries of two congruent Reuleaux triangles that lie in the interior of the other Reuleaux when the intersection corresponds to $(a, b) = (1, 2)$, i.e. according to figure 4.5, by corollary 4.6, we have $1 \leq \dfrac{\left|\widehat{EX}\right| + \left|\widehat{EY}\right|}{\left|\widehat{XY}\right|}$. But the condition that makes equality hold cannot happen when the intersection corresponds to $(a, b) = (1, 2)$, hence the inequality is strict.

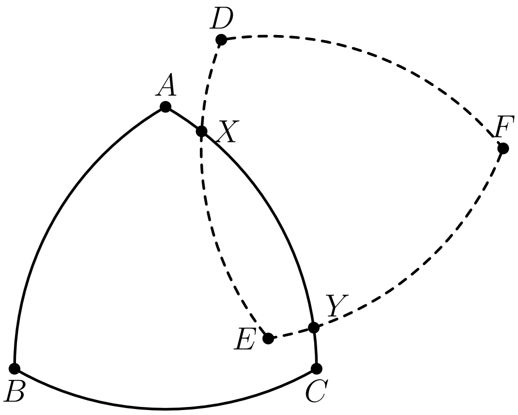
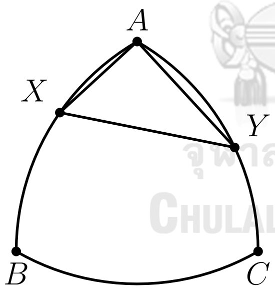


Figure 4.5

The next lemma is an important result that we use to find the upper bound of $\dfrac{\left|\widehat{EX}\right| + \left|\widehat{EY}\right|}{\left|\widehat{XY}\right|}$.

Lemma 4.7.



Let X and Y be two points different from A that lie on arcs $\widehat{AB}$ and $\widehat{AC}$, respectively, of $\mathrm{Reu}\,(ABC)$ *. Then $\left|\overline{AX}\right| \leq \left|\overline{XY}\right|$ (similarly, $\left|\overline{AY}\right| \leq \left|\overline{XY}\right|$).*

Proof. Let Γ be a circle centered at X of radius XA and. Then A is a point of intersection between Γ and the big circle of arc $\widehat{AC}$.

If B, X and A are not collinear, then there is another point of intersection between two circle, say A', as shown in figure 4.6.

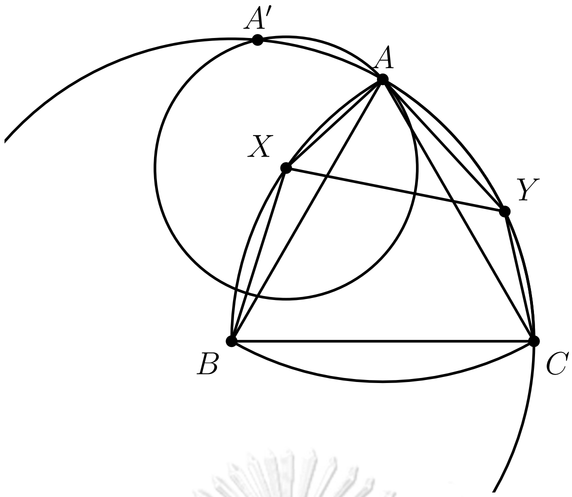


Figure 4.6

Note that A' lie on opposite side of $\overleftrightarrow{BX}$ with $\widehat{AC}$ since $\angle BXA = \frac{5\pi}{6}$. Hence, Y lies outside Γ and, consequently, $|\overline{AX}| \leq |\overline{XY}|$ and the equality holds if and only if $Y = A$ which contradicts our assumption. Hence, in this case, the inequality is strict.

In the case of B, X and A are collinear, this situation can occur only when $X = B$, hence, $|\overline{AX}| = |\overline{AB}| = |\overline{XY}|$ as desired. □

Corollary 4.8. *According to figure* 4.5*, by lemma 4.7 we have* $\left|\widehat{EX}\right| + \left|\widehat{EY}\right| \leq 2\left|\widehat{XY}\right|$*, and the equality holds if and only if* $X = D$ *and* $Y = E$ *which make the intersection does not correspond to* $(a, b) = (1, 2)$*. Hence, the inequality must be strict.*

Now we have a conclusion for Fickett's problem on translation of Reuleaux triangles as follow.

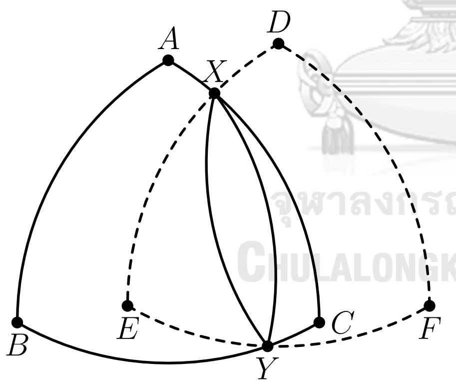
Theorem 4.9 (Main Result 1)**.** *If* R_1 *and* R_2 *are two congruent Reuleaux triangles where* R_2 *is an image of translation of* R_1 *and* $\operatorname{Int}(R_1) \cap \operatorname{Int}(R_2) \neq \varnothing$*, then*

$$\frac{1}{2} < \frac{\operatorname{length}(\partial R_1 \cap \operatorname{Int}(R_2))}{\operatorname{length}(\partial R_2 \cap \operatorname{Int}(R_1))} < 2$$

Moreover, 2 is also the supremum of this ratio, and consequently, by symmetry, $\frac{1}{2}$ *is also the infimum.*

Proof. Using the results from section 4.1, corollaries 4.6 and 4.8, the conclusion is clear when the intersection corresponds to $(a, b) = (1, 2)$.

In the case of the intersection of R_1 and R_2 corrsponds to $(a, b) = (2, 2)$, by propositions 2.1 and 4.4, we can construct two arcs of unit radius connecting X and Y on the interior of intersection area as shown



Note that

$$\frac{\left|\widehat{EX}\right| + \left|\widehat{EY}\right|}{\left|\widehat{CX}\right| + \left|\widehat{CY}\right|} = \frac{\left|\widehat{EX}\right| + \left|\widehat{EY}\right|}{\left|\widehat{XY}\right|} \cdot \frac{\left|\widehat{XY}\right|}{\left|\widehat{CX}\right| + \left|\widehat{CY}\right|}.$$

Hence, by corollaries 4.6 and 4.8, we have

$$\frac{1}{2} = 1 \cdot \frac{1}{2} < \frac{\left|\widehat{EX}\right| + \left|\widehat{EY}\right|}{\left|\widehat{XY}\right|} \cdot \frac{\left|\widehat{XY}\right|}{\left|\widehat{CX}\right| + \left|\widehat{CY}\right|} < 2 \cdot 1 = 2.$$

To show that 2 is the supremum, we consider the following intersection as shown in figure 4.7.

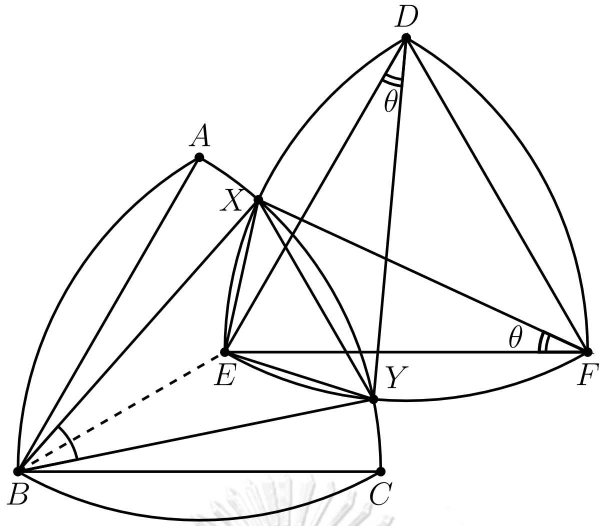


Figure 4.7

Let $\operatorname{Reu}(DEF)$ be a translation of $\operatorname{Reu}(ABC)$ such that $\overrightarrow{BE}$ is the internal bisector of $\angle ABC$. Then, by symmetry, $\angle XFE = \angle YDE$, denote by θ. Note that $0 < \theta < \frac{\pi}{3}$.

Then $\left|\overline{EX}\right| = \left|\overline{EY}\right| = 2\sin\frac{\theta}{2}$ and $\angle XEY = \frac{\pi}{3} + \frac{\pi}{3} - \theta = \frac{2\pi}{3} - \theta$. Hence, $\left|\widehat{XY}\right| = 2\arcsin\left[\cos\left(\frac{\pi}{3} - \theta\right) - \frac{1}{2}\right]$ by using elementary trigonometry, and the ratio can be written as a function of θ as follows.

$$f(\theta) = \frac{\left|\widehat{EX}\right| + \left|\widehat{EY}\right|}{\left|\widehat{XY}\right|} = \frac{\theta}{\arcsin\left(\cos\left(\frac{\pi}{3} - \theta\right) - \frac{1}{2}\right)} \qquad \text{, where } \theta \in \left(0, \frac{\pi}{3}\right).$$

We already know that 2 is an upper bound of $X = \left\{ f(x) \,|\, x \in \left(0, \frac{\pi}{3}\right) \right\}$ and also a limit point of X since $\lim_{x \to \frac{\pi}{3}^-} f(x) = 2$. Hence, 2 is the supremum of X as desired. □

CHAPTER V

FICKETT'S PROBLEM ON GENERAL INTERSECTION OF REULEAUX TRIANGLES

According to chapter 3, we can distinguish all cases of intersection between two congruent Reuleaux triangles.

Note that we have already found the supremum and the infimum of desired ratio in the case of $(a, b) = (1, 1)$, $(1, 2)$ and $(2, 2)$ in section 4 because in the ratio computing step (subsection 4.2), we do not use any special properties of translation. Hence, we can adapt those results from subsection 4.2 to these cases of general intersection, i.e. theorem 4.9 is also suitable for general intersections which correspond to $(a, b) = (1, 1)$, $(1, 2)$ and $(2, 2)$.

But in the case of $(a, b) = (3, 3)$, we found that it is hard to compute the ratio. However, we have found some intersecting result in this case.

Theorem 5.1 (Main Result 2)**.** *If R_1 and R_2 are two Reuleaux triangles of unit width whose intersection corresponds to $(a, b) = (3, 3)$, then the perimeter of intersection area must lie between $\dfrac{2\pi}{3}$ and π.*

Proof. Let $R_1 = \mathrm{Reu}\,(ABC)$ and $R_2 = \mathrm{Reu}\,(DEF)$ be two Reuleaux triangles of unit width whose interior intersect in 6 point as illustrated in figure 5.1

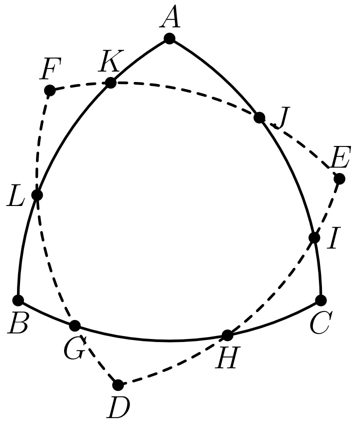


Figure 5.1

Then by lemma 4.7, we have

$$\frac{\pi}{3} = \left|\widehat{BG}\right| + \left|\widehat{GH}\right| + \left|\widehat{HC}\right| < \left|\widehat{LG}\right| + \left|\widehat{GH}\right| + \left|\widehat{HI}\right| \text{ , and}$$
$$\frac{\pi}{3} = \left|\widehat{FK}\right| + \left|\widehat{KJ}\right| + \left|\widehat{JE}\right| < \left|\widehat{LK}\right| + \left|\widehat{KJ}\right| + \left|\widehat{JI}\right|.$$

Combining two above inequalities, we obtain the desired inequality on the left. For the right hand side inequality, by corollary 4.6, we have

$$\begin{aligned}&\left|\widehat{LG}\right| + \left|\widehat{GH}\right| + \left|\widehat{HI}\right| + \left|\widehat{IJ}\right| + \left|\widehat{JK}\right| + \left|\widehat{KL}\right| \\ < \; &\left(\left|\widehat{LB}\right| + \left|\widehat{BG}\right|\right) + \left|\widehat{GH}\right| + \left(\left|\widehat{HC}\right| + \left|\widehat{CI}\right|\right) + \left|\widehat{IJ}\right| + \left(\left|\widehat{JA}\right| + \left|\widehat{AK}\right|\right) + \left|\widehat{KL}\right| \\ = \; &\pi\end{aligned}$$

□

Finally, for further study, we have some claim that might be true after observation for many times as follows.

Claim. *According to the intersection in figure 5.1, for any two Reuleaux triangles* R_1 *and* R_2 *of unit width , if* $\operatorname{length}(\partial R_1 \cap \operatorname{Int}(R_2))$ *is always greater than* $\dfrac{\pi}{3}$*, then the ratio between* $\operatorname{length}(\partial R_1 \cap \operatorname{Int}(R_2))$ *and* $\operatorname{length}(\partial R_2 \cap \operatorname{Int}(R_1))$ *must lies between* $\dfrac{1}{2}$ *and* 2*.*

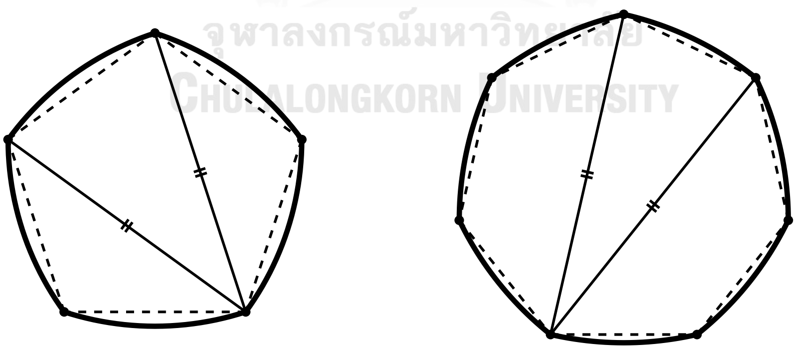
The reason of implication of the claim is if the assumption of the claim is true, i.e. $\operatorname{length}(\partial R_1 \cap \operatorname{Int}(R_2)) > \dfrac{\pi}{3}$, we also obtain that $\operatorname{length}(\partial R_2 \cap \operatorname{Int}(R_1)) > \dfrac{\pi}{3}$ by symmetry and hence by theorem 5.1 we have

$$\frac{\pi}{3} < \operatorname{length}(\partial R_1 \cap \operatorname{Int}(R_2)) < \frac{2\pi}{3} \quad \text{and} \quad \frac{\pi}{3} < \operatorname{length}(\partial R_2 \cap \operatorname{Int}(R_1)) < \frac{2\pi}{3}.$$

Consequently, these two inequalities imply that

$$\frac{1}{2} < \frac{\operatorname{length}(\partial R_1 \cap \operatorname{Int}(R_2))}{\operatorname{length}(\partial R_2 \cap \operatorname{Int}(R_1))} < 2.$$

Moreover, the Fickett's problem for another convex curves of constant width, e.g. Reuleaux $n-$gon where $n \geq 3$ is odd, is very interesting for generalization in further study.



(a) (b)

Figure 5.2: Reuleaux 5−gon and Reuleaux 7−gon are illustrated in figure 5.2a and 5.2b, respectively.

REFERENCES

[1] Bartle, R. G., and Sherbert, D. R. *Introduction to Real Analysis*, 4^{th} ed., Hoboken, NJ: Wiley, 2011.

[2] Croft, H., Falconer, K. and Halmos, P. *Unsolved Problems in Geometry.*, New York: Springer New York, 1991.

[3] Fickett, J.W. Overlapping congruent convex bodies. *The American Mathematical Monthly* **87** (December 1980) : 814-815.

[4] Nielsen, C. and Powers, C. Intersecting equilateral triangles. *Forum Geometricorum* **13** (2013) : 219–225.

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