Chapter II
Theoretical Background

The characteristics of the acceleration of particles on the Sun or in the corona are not certain. A major goal of this thesis research is involved with the injection of particles from the Sun, so it is necessary to consider the influences that affect the transport. We need to select the appropriate equation for this propagation. The author uses the theory of focused transport to explain the propagation of solar cosmic rays in the interplanetary space, and to consider the effects described in this chapter.

The Solar Wind

The outermost portion of the Sun is the corona. It has a high temperature ($\sim 10^6$K), so it has a higher pressure than the surrounding interplanetary medium. The pressure in the interplanetary medium is close to zero because the density of plasma is lower. Thus the pressure difference forces a flow from the corona to the interplanetary medium. This difference drives the plasma to a high velocity, so there has been a change of energy from thermal energy to kinetic energy. Plasma continuously flows away from the Sun with a speed of ≈ 400 km/s and this flow is called the "solar wind." This flow "drags" the magnetic field out from the Sun (this "dragging" will be explained in the next section). Since the solar wind is highly turbulent, the interplanetary magnetic field which is dragged out is very irregular (Figure 2.1).

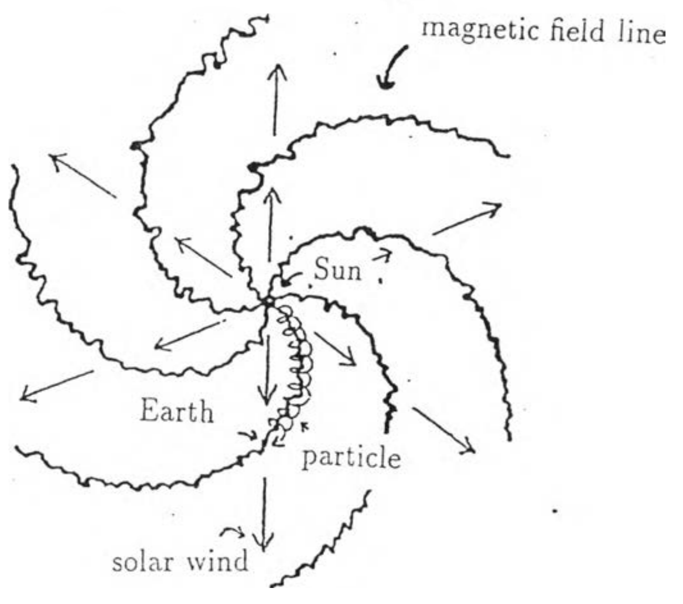

Figure 2.1: Sketch of the solar wind and the interplanetary magnetic field.

Interplanetary Magnetic Field

The particles released from the Sun move in the interplanetary medium, which comprises a plasma continuously moving with a speed of ≈ 400 km/s (the solar wind). The solar wind carries the magnetic field from the Sun, but the Sun turns around on its axis, so the magnetic field lines are bent as in Figure 2.1. The turbulent flow of the solar wind leads to irregular magnetic field lines. The transport of particles in this magnetic field mainly consists of motion around the magnetic field line.

To study the propagation of solar cosmic rays requires understanding the interplanetary magnetic field. Because of the strength of the tenuous interplanetary plasma (< 10 particles/cm^3 at 1 AU), the interplanetary magnetic field is convected out from the Sun. This is because it is "frozen" into the plasma. Consider the magnetic flux, Φ, through any surface, S, bounded by the closed contour, L (Figure 2.2). After a small time, Δt, let L be convected with the plasma velocity, $\vec{u}$, to a new contour, L'. Let S' be any surface bounded by L'.

Then the comoving time derivative of the flux is given by

$$\begin{aligned}
\frac{d\Phi}{dt} &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(\int_{s'}\vec{B}(t+\Delta t)\cdot d\vec{a}' - \int_{s}\vec{B}(t)\cdot d\vec{a}\right)\\
\int_{s'}\vec{B}(t+\Delta t)\cdot d\vec{a}' - \int_{s}\vec{B}(t)\cdot d\vec{a} &= \int_{s'}\vec{B}(t+\Delta t)\cdot d\vec{a}' - \int_{s'}\vec{B}(t)\cdot d\vec{a}'\\
&\quad + \int_{s'}\vec{B}(t)\cdot d\vec{a}' - \int_{s}\vec{B}(t)\cdot d\vec{a}
\end{aligned}$$

Using Green's theorem, Stokes's theorem, and Ampère's Law,

$$\begin{aligned}
\int_{s'}\vec{B}(t+\Delta t)\cdot d\vec{a}' - \int_{s}\vec{B}(t)\cdot d\vec{a}' &= \Delta t\int_{s'}\frac{\partial\vec{B}}{\partial t}\cdot d\vec{a}' + \int_{v}\vec{\nabla}\cdot\vec{B}dV - \int_{s''}\vec{B}\cdot d\vec{a}''\\
&= -\Delta t\int_{s'}c(\vec{\nabla}\times\vec{E})\cdot d\vec{a}' + 0\\
&\quad + \oint_{L'}\vec{B}\cdot(\vec{u}\Delta t\times d\vec{l}') \qquad \text{as } (\Delta t\to 0)\\
&= -\Delta t\oint_{L'}c\vec{E}\cdot d\vec{l}' - \oint_{L'}(\vec{u}\Delta t\times\vec{B})\cdot d\vec{l}'\\
\frac{d\Phi}{dt} &= \lim_{\Delta t\to 0} c\oint_{L'}\left(\vec{E}+\frac{1}{c}\vec{u}\times\vec{B}\right)\cdot d\vec{l} \qquad (2.1)
\end{aligned}$$

Now $\vec{E}_r = \gamma\left(\vec{E}-(1/c)\vec{u}\times\vec{B}\right)$ where $\vec{E}_r$ is the electric field in the rest frame of the plasma and $\gamma = (1-u^2/c^2)^{-1/2}$. However, if the plasma is perfectly conducting, $\vec{E}_r \equiv 0$. Therefore, we have $d\Phi/dt \equiv 0$. This implies that magnetic field lines are dragged along with the plasma (Roelof 1969).

The concept of being "frozen-in" means that the magnetic field can be deduced from the plasma velocity field. The solar plasma initially corotates with the surface of the Sun.

Assuming that the plasma flows radially out from the Sun, the mean field $B_0(\theta)$ is $B(r_0,\theta)$, where θ is measured from the north ecliptic pole. The magnetic field lines lie along the Archimedean spirals $\phi = \phi_0 - \Omega r\sin\theta/u$. We have

$$\Delta r = u\Delta t,\ \Delta\phi = -\Omega\Delta t,\ \frac{dr}{d\phi} = -\frac{u}{\Omega}. \qquad (2.2)$$

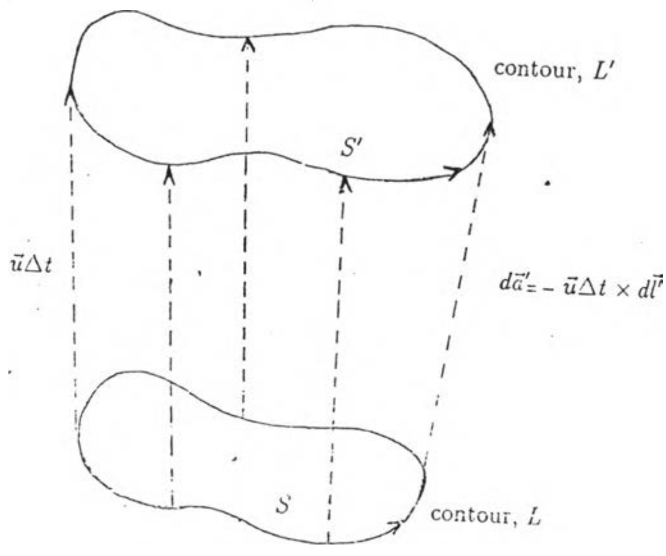

Figure 2.2: The change of flux through any closed contour, L, is zero in a perfectly conducting plasma (after Roelof 1969).

We define $\phi = 0$ when $r = 0$, so $d\phi/dt = 0$, and

$$\begin{aligned} \vec{B}(r,\theta) &= a(r,\theta)\left(\vec{e}_r + \frac{r\sin\theta d\phi}{dr}\vec{e}_\phi\right), \\ &= a(r,\theta)\left(\vec{e}_r - \frac{\Omega r\sin\theta}{u}\vec{e}_\phi\right). \end{aligned} \tag{2.3}$$

We must have $\nabla \cdot \vec{B} = 0$, so

$$\begin{aligned} 0 &= \frac{1}{r^2}\frac{\partial}{\partial r}r^2 B_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta B_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}B_\phi, \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}r^2 a, \\ a &= \frac{f(\theta)}{r^2}. \end{aligned} \tag{2.4}$$

If $\vec{B}_0(\theta) \equiv \vec{B}(r_0,\theta)$ then

$$\begin{aligned} f(\theta) &= r_0^2 B_{0r}(\theta) = -\frac{ur_0}{\Omega\sin\theta}B_{0\phi}(\theta) \\ \text{and } B_{0\phi} &= -\frac{\Omega r_0\sin\theta}{u}B_{0r}. \end{aligned} \tag{2.5}$$

If we use the values of $\Omega = 3\times 10^{-1}\ \text{sec}^{-1}$, $r_0 \approx 700,000$ km, and $u \approx 400$ km/sec, then $B_{0\phi} \approx -0.005\ B_{0r}$, so $B_0(\theta) \approx \sqrt{(B_{0r})^2 + (B_{0\phi})^2} \approx B_{0r}(\theta)$. Finally, the field at a distance r from the Sun is

$$B(r,\theta) = B_0(\theta)\left(\frac{r_0}{r}\right)^2\left(\vec{e}_r - \frac{\Omega r\sin\theta}{u}\vec{e}_\phi\right) \tag{2.6}$$

where r_0 is the radius of the Sun, Ω is the angular rate of rotation of the Sun, and $\vec{e}_r$ and $\vec{e}_\phi$ are unit vectors in the radial and azimuthal directions, respectively (Roelof 1969). Such a field line corotates with the Sun, and it has major effects on the propagation of energetic particles spiraling along it.

Charged Particle Transport

As mentioned earlier, the turbulence of the solar wind makes the magnetic field from the Sun very irregular. This characteristic affects the charged particle motion. When we consider the transport of charged particles in the magnetic field, we find that the orbit of a particle is a helix. because particles are forced to move in circles around the magnetic line, while moving along the field with a constant speed (Figure 2.3).

The form of this orbit depends on the pitch angle (θ), which is the angle between the speed of the particle and the magnetic field line. Defining

$$\mu \equiv \cos\theta, \tag{2.7}$$

μ is a constant of the motion for a uniform magnetic field. However, in the interplanetary medium, it is affected by the irregularities in the magnetic field. The sign of μ will show the direction of transport from the Sun. When $\theta < 90°$, $\mu > 0$, then the particles move outward from the Sun, and when $\theta > 90°$, $\mu < 0$, then the particles move toward the Sun. If the magnetic field is smooth, then the form of this orbit will be constant, but when the interplanetary magnetic field is irregular, then the value of μ changes randomly. This process is called "pitch angle scattering" and is shown in Figure 2.4.

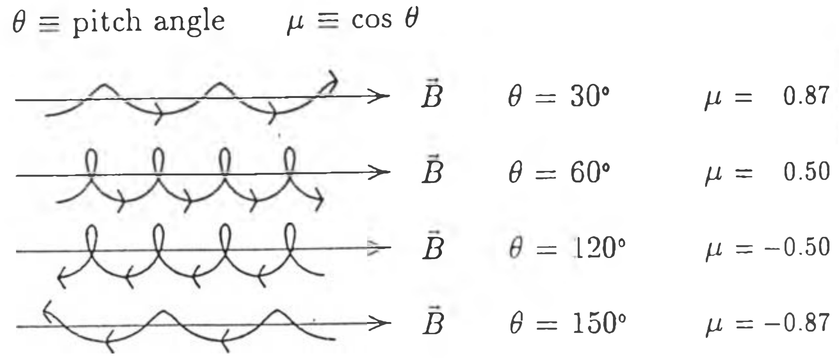

Figure 2.3: Pitch-angle scattering in a steady magnetic field.

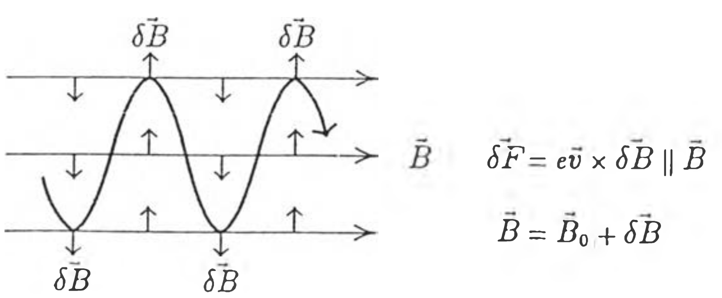

Figure 2.4: Example of the force along the field for resonant scattering of particles orbiting an irregular magnetic field.

The simplest treatment of the diffusive transport of particles in interplanetary space includes the effects of the streaming of particles and the distribution of scattering. Jokipii (1966) used a Fokker-Planck equation to explain this transport of particles, using a Fokker-Planck equation of the form

$$\frac{\partial f(t,\mu,z)}{\partial t} = -\mu v \frac{\partial f(t,\mu,z)}{\partial z} + \frac{\partial}{\partial \mu}\frac{\varphi(\mu)}{2}\frac{\partial f(t,\mu,z)}{\partial \mu}, \tag{2.8}$$

where t is the time since the solar flare occurrence, z is the arclength along the magnetic field, v is the particle speed, μ is the cosine of the pitch angle, or v_z/v, f is the particle distribution function and φ is the coefficient of pitch angle scattering (Jokipii 1971; Earl 1973), where

$$\varphi(\mu) = A|\mu|^{q-1}(1-\mu^2). \tag{2.9}$$

This equation includes the effects of pitch-angle scattering and streaming, which explain the cosmic-ray distribution as a function of the distance along the magnetic field and the cosine of the pitch angle, in which q is the spectral index of the power law for interplanetary field fluctuations at wave number k within an interval dk, Q_{xx} is the spectral power at a reference wave number k_0, and

$$A = 2\pi\frac{v}{R^2}Q_{xx}(k_0 r_L)^q, \tag{2.10}$$

where R is the particle rigidity, and r_L is the Larmor radius. Thus Q_{xx}, q, and k_0 are parameters of the spectrum of field irregularities (Jokipii 1966).

Earl (1976a) further developed the transport equation by including the effects of adiabatic focusing:

$$\frac{\partial f(t,\mu,z)}{\partial t} + \mu v\frac{\partial f(t,\mu,z)}{\partial z} = -\frac{v}{2L(z)}(1-\mu^2)\frac{\partial f(t,\mu,z)}{\partial \mu} + \frac{\partial}{\partial \mu}\frac{\varphi(\mu)}{2}\frac{\partial f(t,\mu,z)}{\partial \mu} \tag{2.11}$$

where the first term on the right-hand side represents the effect of adiabatic focusing (Roelof 1969), and L is the scale length for spatial variations of the guiding field,

$$\frac{1}{L(z)} = -\frac{1}{B}\frac{\partial B}{\partial z}. \tag{2.12}$$

In this work, the author uses the focused transport equation as the basis for simulating coherent pulses of solar cosmic rays. However, this is not the most complete equation used in this work, which will include other effects.

Adiabatic Deceleration

In the focused transport model (Earl 1976a), the energetic particles are considered to undergo pitch-angle scattering, which is the effect of small-scale irregularities in the interplanetary magnetic field, and focusing, which is the effect of the large-scale divergence of the field at increasing distance from the Sun. We can then consider two reference frames (Ruffolo 1995). The first is the fixed frame (we define the particle velocity in this frame as $\vec{v}$). In the fixed frame, the large-scale structure of the magnetic field is stationary, and the focusing conserves $v = |\vec{v}|$ (Figure 2.5a). The second frame is the solar wind frame (we define the particle velocity in this frame as $\vec{v}'$). In the solar wind frame, the small-scale irregularities are frozen in the solar wind frame, so the scattering conserves the magnitude of the velocity, $v' = \left|\vec{v}'\right|$ (Figure 2.5b). We examine effects of focusing and scattering in the solar wind frame, because it is computationally easier to consider scattering in the frame in which it conserves the particle speed.

Focusing preserves v but does not preserve v'. The focusing always makes the velocity of the particle in the solar wind frame closer to the origin. Figure 2.6 shows the deceleration of v' from the schematic trajectory of a particle that undergoes scattering and focusing.

Note that the rate of deceleration is

$$\dot{v}' = \left.\frac{dv'}{d\mu}\right|_{v} \dot{\mu}, \tag{2.13}$$

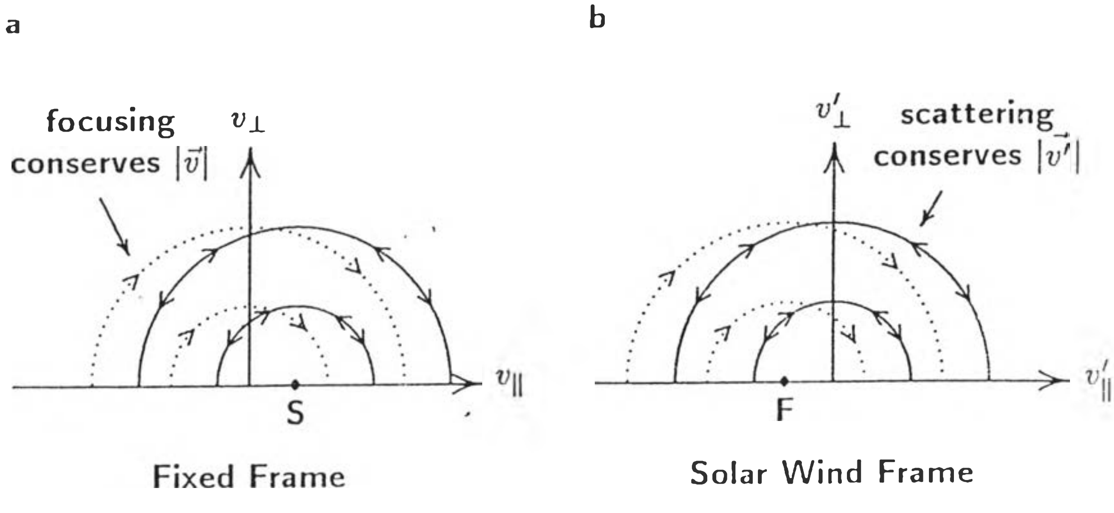

Figure 2.5: Illustration of the adiabatic focusing and pitch-angle scattering in (a) the fixed frame and (b) the solar wind frame.

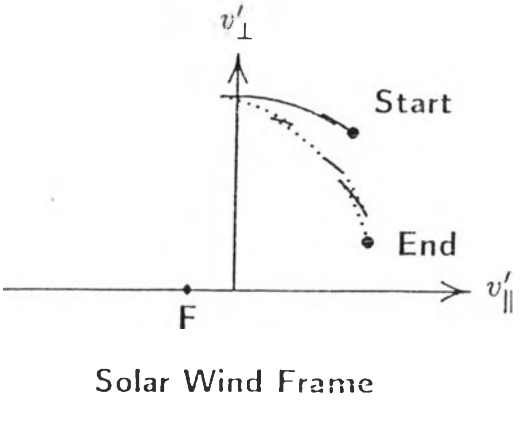

Figure 2.6: Scattering and focusing in the solar wind frame.

and the focusing in the fixed frame gives the rate of change of μ as

$$\dot{\mu} = \frac{v}{2L(z)}(1 - \mu^2), \tag{2.14}$$

where the scale length, L, for spatial variation of the magnetic field was defined earlier. Substituting $\dot{\mu}$ from equation (2.14) into equation (2.13), we find that

$$\begin{aligned}
v' &= \sqrt{v_{||}'^2 + v_{\perp}'^2} \\
v_{||}' &= v_{||} - v_{sw} \\
\sqrt{1 - \mu'^2}\; v' = v_{\perp}' &= v_{\perp} = \sqrt{1 - \mu^2}\; v \\
v' &= \sqrt{(v_{||} - v_{sw})^2 + v_{\perp}^2} \\
&= (v^2 - 2\mu v v_{sw} + v_{sw}^2)^{1/2} \\
\left.\frac{dv'}{d\mu}\right|_v &= -v_{sw} v / v' \\
\dot{v}' &= \frac{-v_{sw}}{2L(z)}\frac{v^2}{v'}(1 - \mu^2) \\
&= \frac{-v_{sw} v'}{2L(z)}(1 - \mu'^2),
\end{aligned} \tag{2.15}$$

where v_{sw} is the solar wind velocity. Similarly, if we consider special relativity, and neglect terms of order $(v_{sw}/c)^2$, we get the formula for the rate of change of the momentum,

$$\dot{p}' = -\frac{v_{sw} p'}{2L(z)}(1 - \mu'^2), \tag{2.16}$$

where p' is the momentum of a particle in the solar wind frame.

For a radial field, the scale length of the interplanetary field is $r/2$, where r is the distance from the Earth to the Sun, so we have

$$\frac{v_{sw}}{L(z)} = \frac{2v_{sw}}{r} = \nabla \cdot v_{sw}. \tag{2.17}$$

From equations (2.16) and (2.17), we get

$$\dot{p}' = \frac{p'}{2}(1 - \mu'^2)\nabla \cdot v_{sw}. \tag{2.18}$$

In the solar wind frame, the pitch-angle distribution tends to become isotropic, in which case the directional average of $(1-\mu'^2)$ is equal to 2/3 and

$$\langle \dot{p}' \rangle = -\frac{p'}{3}\nabla \cdot v_{sw}. \tag{2.19}$$

From this viewpoint, adiabatic deceleration is a monotonic decrease in the momentum resulting from the transformation of adiabatic focusing from the fixed frame to the solar wind frame (Ruffolo 1995).

Archimedean Spiral Magnetic Field

In a frame that is corotating with the Sun, the solar wind velocity, v^c_{sw}, is parallel to the magnetic field at each point:

$$\vec{v}^c_{sw} = v_{sw}\hat{r} - \Omega r \sin\theta\hat{\varphi} \tag{2.20}$$

$$= v^c_{sw}\hat{z} \tag{2.21}$$

$$= v_{sw}\sec[\psi(z)], \tag{2.22}$$

where Ω is the angular velocity of the solar rotation, $\hat{z}$ is the unit vector along the outward tangent to the average magnetic field, and $\psi(z)$ is the angle between $\hat{r}$ and $\hat{z}$, which is shown in Figure 2.7.

The decrease of p' is systematic, and the rate of change in the magnitude of the momentum of a particle in the solar wind frame depends on the adiabatic focusing. By replacing v^c_{sw} for v_{sw} in equation (2.16), we get

$$\dot{p}' = \frac{-v^c_{sw}p'}{2L(z)}(1-\mu'^2). \tag{2.23}$$

The rate of change of momentum depends on the change in velocity along the magnetic field (v^c_{sw}) in the solar wind frame. In different locations, the solar wind frame and v^c_{sw} are different. This characteristic is called "differential convection."

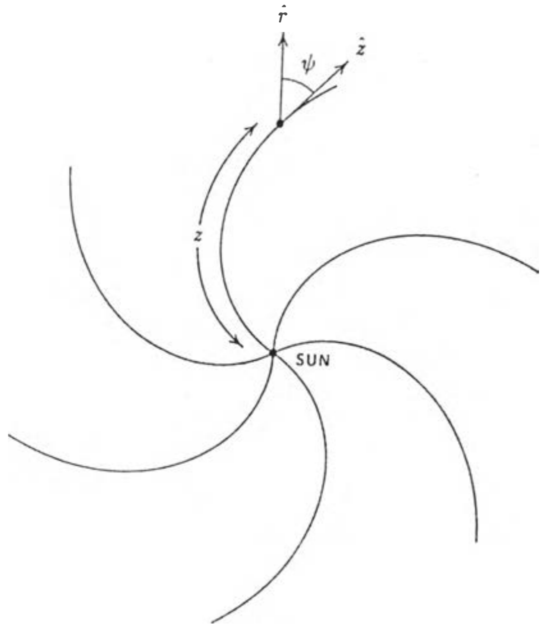

Figure 2.7: Illustration of the direction of $\hat{r}$, $\hat{z}$, and $\psi(z)$.

If we consider the streaming of a particle moving from point A to B along the same field line in the solar wind frame, focusing preserves v and increases $p_{||} = p'_{||} + E/(c^2)v^c_{sw}$ or $p'_{||} = p_{||} - E/(c^2)v^c_{sw}$, so we get the relation between p'_A, the momentum in the solar wind frame at point A, and p'_B, that at point B:

$$p'_{||,A} = p_{||} - \frac{E}{c^2}v^c_{sw,A} \qquad (2.24)$$

$$p'_{||,B} = p_{||} - \frac{E}{c^2}v^c_{sw,B}, \qquad (2.25)$$

and the momentum change from point A to B is

$$\Delta p'_{||} = -\frac{E}{c^2}v_{sw}(\Delta \sec\varphi). \qquad (2.26)$$

In terms of the distance along the field line from A to B, this is

$$\Delta p'_{||} = -\frac{E}{c^2}v_{sw}\left(\frac{d}{dz}\sec\psi\right)\Delta z. \qquad (2.27)$$

From $\Delta z = v_{||}\Delta t$,

$$\Delta p'_{||} = -\frac{E}{c^2}v_{sw}\left(\cos\phi\frac{d}{dr}\sec\phi\right)v_{||}\Delta t, \qquad (2.28)$$

where Δt is the travel time. Finally, the rate of deceleration along the field line due to this effect is

$$\dot{p}'_{\|} = -p_{\|} v_{sw} \left(\cos\psi \frac{d}{dr} \sec\psi \right). \tag{2.29}$$

From the relation, $\dot{p}' = (p'_{\|}/p')\dot{p}_{\|}'$, we can find the rate of deceleration due to changes in v^c_{sw}:

$$\dot{p}' = -p' v_{sw} (\cos\psi \frac{d}{dr} \sec\psi) \mu'^2. \tag{2.30}$$

Therefore from equation (2.23) and equation (2.30), we get a total deceleration rate of

$$\dot{p}' = -p' v_{sw} \left(\frac{\sec\psi}{2L(z)}(1 - \mu'^2) + \cos\psi \frac{d}{dr} \sec\psi \mu'^2 \right). \tag{2.31}$$

Other Effects of the Solar Wind

There are other effects of the solar wind on the transport of solar flare particles. Due to the effects of scattering, the distribution of particles is often nearly isotropic in the solar wind frame. However, in the fixed frame, solar wind convection makes the distribution anisotropic. Since focusing preserves v in the fixed frame, and $p_{\|} = p'_{\|} + E/(c^2) v^c_{sw}$, then

$$p_{\|} = \mu' p' + \frac{E}{c^2} v_{sw} \sec\psi. \tag{2.32}$$

The rate of streaming and convection is given by

$$\begin{aligned} \dot{z} &= v_{\|} = \frac{E'}{E} \mu' v' + v_{sw} \sec\psi & (2.33) \\ &= \mu' v' + \left(1 - \mu'^2 \frac{v'^2}{c^2} \right) v_{sw} \sec\psi. & (2.34) \end{aligned}$$

The change in μ is a result from adiabatic focusing according to equation (2.14) in the fixed frame, so in the solar wind frame this rate becomes

$$\dot{\mu}' = \left. \frac{d\mu'}{d\mu} \right|_p \dot{\mu} = \frac{v'}{2L(z)} \left(1 + \mu' \frac{v_{sw}}{v'} \sec\psi - \mu' v_{sw} \frac{v'}{c^2} \sec\psi \right) (1 - \mu'^2) \tag{2.35}$$

where $\dot{\mu}'$ is differential convection from the changes of the solar wind velocity v_{sw}^c. The perpendicular momentum in the solar wind frame is not affected by this, so we have

$$d(p_\perp'^2) = 0 \quad = \quad d[p'^2(1 - \mu'^2)] \tag{2.36}$$

$$= \quad 2(1 - \mu'^2)p'dp' - 2\mu'p'^2 d\mu' \tag{2.37}$$

$$\dot{\mu}' \quad = \quad \frac{1 - \mu'^2}{\mu'} \frac{\dot{p}'}{p'}. \tag{2.38}$$

Considering this equation (2.38) and equation (2.30) we get

$$\dot{\mu}' = -v_{sw}(\cos\psi \frac{d}{dr} \sec\psi)\mu'(1 - \mu'^2). \tag{2.39}$$

From equation (2.35) and equation (2.39), we find the total rate of change of μ from the effects of the solar wind and due to focusing:

$$\begin{aligned} \dot{\mu}' \quad = \quad & \frac{v'}{2L(z)} \left[1 + \mu' \frac{v_{sw}}{v'} \sec\psi - \mu' \frac{v_{sw} v'}{c^2} \sec\psi\right] (1 - \mu'^2) \\ & - v_{sw}(\cos\psi \frac{d}{dr} \sec\psi)\mu'(1 - \mu'^2). \end{aligned} \tag{2.40}$$

This differs from equation (2.35) because we are considering the other effects in the theory of focused transport for the propagation of solar cosmic rays in the interplanetary space. Now we can find the appropriate equation for this propagation.

Modified Equation of Focused Transport

Now we have new expressions as a function of the pitch angle from equation (2.40), the distance from the Sun along the magnetic field from equation (2.34) and the momentum from equation (2.31). From these equations, we can find a Fokker-Planck equation for solar cosmic rays in interplanetary space, which will be an improved form of equation (2.8). We give the distribution function of solar cosmic rays F as

$$F(t, \mu, z, p) = \frac{d^3N}{dz d\mu dp}, \tag{2.41}$$

where N represents the number of particles. The Fokker-Planck equation that includes only the effects of streaming, adiabatic focusing and scattering is

$$\frac{\partial F(t,\mu,z,p)}{\partial t} = -\mu v \frac{\partial F(t,\mu,z,p)}{\partial z} - \frac{v}{2L}\frac{\partial}{\partial \mu}(1-\mu^2)F(t,\mu,z,p) + \frac{1}{2}\frac{\partial}{\partial \mu}\varphi(\mu)\frac{\partial F(t,\mu,z,p)}{\partial \mu}. \tag{2.42}$$

We write this equation in terms of changes of p, μ, z in the local solar wind frame (without primes, for convenience):

$$\begin{aligned}\frac{\partial F(t,\mu,z,p)}{\partial t} &= -\frac{\partial}{\partial z}\left(\frac{\Delta z}{\Delta t}F\right) - \frac{\partial}{\partial \mu}\left(\frac{\Delta \mu}{\Delta t}F\right) \\ &+\frac{\partial}{\partial \mu}\left[\frac{\varphi(\mu)}{2}\frac{\partial}{\partial \mu}\left(\frac{E'}{E}F\right)\right] - \frac{\partial}{\partial p}\left(\frac{\Delta p}{\Delta t}F\right).\end{aligned} \tag{2.43}$$

Note that there is a factor of $E'/E = 1 - \mu v v_{sw} \sec\psi / c^2$ in the pitch-angle scattering term. It relates to the distribution in terms of time and position in the local solar wind frame (Webb & Glesson 1979; Skilling 1975; Earl 1984). In equations (2.31), (2.34) and (2.40) we presented $\Delta z/\Delta t$, $\Delta \mu/\Delta t$ and $\Delta p/\Delta t$, so we can derive the appropriate transport equation (Ruffolo 1995):

$$\begin{aligned}\frac{\partial F(t,\mu,z,p)}{\partial t} &= -\frac{\partial}{\partial z}\mu v F(t,\mu,z,p) && \text{(streaming)} \\ &-\frac{\partial}{\partial z}\left(1-\mu^2\frac{v^2}{c^2}\right)v_{sw}\sec\psi F(t,\mu,z,p) && \text{(convection)} \\ &-\frac{\partial}{\partial \mu}\frac{v}{2L(z)}\left[1+\mu\frac{v_{sw}}{v}\sec\psi - \mu\frac{v_{sw}v}{c^2}\sec\psi\right] \\ &\cdot(1-\mu^2)F(t,\mu,z,p) && \text{(focusing)} \\ &+\frac{\partial}{\partial \mu}v_{sw}\left(\cos\psi\frac{d}{dr}\sec\psi\right)\mu(1-\mu^2) \\ &\cdot F(t,\mu,z,p) && \text{(differential convection)} \\ &+\frac{\partial}{\partial \mu}\frac{\varphi(\mu)}{2}\frac{\partial}{\partial \mu}F(t,\mu,z,p) && \text{(scattering)} \\ &+\frac{\partial}{\partial p}p v_{sw}\left[\frac{\sec\psi}{2L(z)}(1-\mu^2)+\cos\psi\frac{d}{dr}\sec\psi\mu^2\right] \\ &\cdot F(t,\mu,z,p) && \text{(deceleration)}\end{aligned} \tag{2.44}$$

The parameters in this equation are the angle between the field line and the radial direction $\psi(z)$, the focusing length $L(z)$, and the pitch-angle scattering coefficient $\varphi(\mu)$. From the Archimedean field model of Parker (1958) with $b = 0$, we get the relation

$$\begin{aligned}
\frac{1}{L} &\equiv -\frac{1}{B}\frac{dB}{dz} = -\frac{1}{B}\frac{dr}{dz}\frac{dB}{dr} && (2.45)\\
B &\propto \frac{R}{r} + \frac{R^2}{r^2} && (2.46)\\
\frac{dz}{dr} &= \sec\psi = \sqrt{1+\left(\frac{r}{R}\right)^2} && (2.47)\\
\cos\psi &= \frac{v_{sw}}{\sqrt{v_{sw}^2 + \Omega^2 r^2 \sin^2\theta}} = \frac{R}{\sqrt{r^2+R^2}} && (2.48)\\
L &= \frac{r(r^2+R^2)^{3/2}}{R(r^2+2R^2)}, && (2.49)
\end{aligned}$$

where $R = v_{sw}/(\Omega\sin\theta)$ is the angular rotation rate of the Sun, and r is the radius as a function of z. We use the coefficient of pitch-angle scattering as

$$\varphi(\mu) = A|\mu|^{q-1}(1-\mu^2) \qquad (2.50)$$

There is a problem from the singularity at μ=0, so we follow Ng & Wong (1979) and use an effective scattering coefficient $\varphi_{\text{eff}}(\mu)$. To derive $\varphi_{\text{eff}}(\mu)$, we start with the μ flux,

$$\begin{aligned}
S_\mu &= \frac{v}{2L}(1-\mu^2)F - \frac{\varphi(\mu)}{2}\frac{\partial F}{\partial \mu}\\
&= \frac{v}{2L}(1-\mu^2)F - \frac{A}{2}|\mu|^{q-1}(1-\mu^2)\frac{\partial F}{\partial \mu} && (2.51)
\end{aligned}$$

assume that $S_\mu = s(1-\mu^2)$ for simplicity. Then we consider $F(\mu)$ at fixed t, z and μ:

$$\begin{aligned}
s(1-\mu^2) &= \frac{v}{2L}(1-\mu^2)F(\mu) - \frac{A}{2}|\mu|^{q-1}(1-\mu^2)\frac{d}{d\mu}F(\mu)\\
0 &= \left(F(\mu) - \frac{2Ls}{v}\right) - \frac{AL}{v}|\mu|^{q-1}\frac{d}{d\mu}\left(F(\mu) - \frac{2Ls}{v}\right)\\
\int \frac{d(F(\mu) - 2Ls/v)}{(F(\mu) - 2Ls/v)} &= \int \frac{v}{AL|\mu|^{q-1}}d\mu\\
\ln(F(\mu) - 2Ls/v) + c' &= \frac{v}{AL}\int |\mu|^{1-q}d\mu\\
F(\mu) &= 2Ls/v + ce^{\operatorname{sgn}(\mu)\frac{v}{AL}\frac{|\mu|^{2-q}}{(2-q)}} && (2.52)
\end{aligned}$$

We find the constant values of s and c when $\mu = \mu_i$, μ_{i+1} and $\mu_i = \overline{\mu} - \Delta\mu/2$, $\mu_{i+1} = \overline{\mu} + \Delta\mu/2$.

$$
\begin{aligned}
c &= \frac{\Delta F}{2e^{\frac{v}{2AL(2-q)}(|\overline{\mu}-\Delta\mu/2|^{2-q}+|\overline{\mu}+\Delta\mu/2|^{2-q})}} \\
&\quad \cdot \sinh\left(\frac{v}{2AL(2-q)}(-|\overline{\mu}-\Delta\mu/2|^{2-q}+|\overline{\mu}+\Delta\mu/2|^{2-q})\right) \\
s &= \frac{v}{AL}\overline{F} - \frac{v}{4L}\Delta F \coth\left(\frac{v}{2AL(2-q)}(|\overline{\mu}+\Delta\mu/2|^{2-q}-|\overline{\mu}-\Delta\mu/2|^{2-q})\right)
\end{aligned}
$$

where $\overline{F} \equiv (F_i + F_{i+1})/2$, $\Delta F \equiv F_{i+1} - F_i$ so

$$
\begin{aligned}
S_\mu(\overline{\mu}) &= \frac{v}{2L}(1-\overline{\mu}^2)\overline{F} - \frac{\varphi_{\text{eff}}(\overline{\mu})}{2}\frac{\Delta F}{\Delta\mu} \\
\varphi_{\text{eff}}(\mu) &= \frac{v}{2L(z)}(1-\mu^2)\cdot \\
&\quad \frac{\Delta\mu}{\tanh\{v[I(\mu+\Delta\mu/2)-I(\mu-\Delta\mu/2)]/[2AL(z)]\}}.
\end{aligned}
\tag{2.53}
$$

where

$$
\begin{aligned}
\frac{dI(\mu)}{d\mu} &= |\mu|^{1-q}, \\
I(\mu) &= \text{sgn}(\mu)\frac{|\mu|^{2-q}}{2-q}.
\end{aligned}
\tag{2.54}
$$

The limit of the effective scattering coefficient when $\Delta\mu$ is close to 0 is

$$
\begin{aligned}
\lim_{\Delta\mu\to 0} &= \frac{(v/2L)\Delta\mu(1-\mu^2)}{(v/2AL)|\mu|^{1-q}\Delta\mu} \\
&= A|\mu|^{q-1}(1-\mu^2).
\end{aligned}
\tag{2.55}
$$

where $\Delta\mu$ is the grid spacing in the μ coordinate. We thus find that $\varphi_{\text{eff}}(\mu) \to \varphi(\mu)$ when $\mu \gg \Delta\mu$.

The equation for the pitch-angle scattering in terms of a spatial mean free path, λ, is

$$
\lambda \equiv \frac{3D}{v}, \tag{2.56}
$$

where D is a spatial diffusion coefficient, and

$$D = \frac{v^2}{4}\int_{-1}^{1}\frac{(1-\mu^2)^2}{\varphi(\mu)}d\mu \tag{2.57}$$

(Hasselmann and Wibberenz 1968). In the numerical code, we use

$$\lambda = \frac{3D}{V} = \frac{3v}{4}\sum_{\mu}\frac{(1-\mu^2)^2}{\varphi_{\text{eff}}(\mu)}\Delta\mu \tag{2.58}$$

in the limit of no focusing (Ruffolo 1991), where the sum is over μ halfway between grid points. We can find the scattering amplitude, A, that leads to the desired value of λ.