CHAPTER VI
PRISM(S_n)

In this chapter, we show that the prism of star S_n is an edge-odd graceful graph for every $n \geq 3$.

Definition 6.1 Let $n \geq 3$ and S_n be a star. Let S_n' be a copy of S_n. Define $\text{Prism}(S_n)$, called the **prism of** S_n, by joining u of S_n to the corresponding vertex u' of S_n' and each u_i of S_n to the corresponding vertex u_i' of S_n' for all $i \in \{1, 2, 3, \dots, n\}$. Thus,

$$E(\text{Prism}(S_n)) = E(S_n) \cup E(S_n') \cup \{u_i u_i' \mid i \in \{1, 2, 3, \dots, n\}\} \cup \{uu'\}.$$

Example 6.1 From Definition 6.1, we have $\text{Prism}(S_4)$, shown in Figure 6.1.

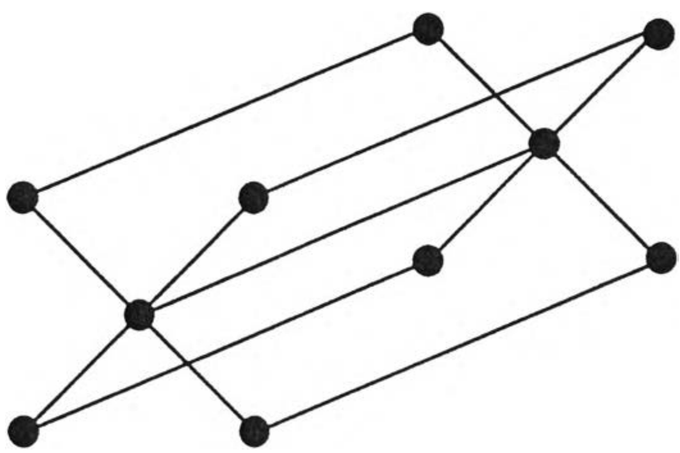

Figure 6.1 $\text{Prism}(S_4)$.

Algorithm 6.1

Let $n > 3$ be an even integer. Let G denote $\text{Prism}(S_n)$. Then, $q = 3n + 1$.

Define $f: E(G) \to \{1, 3, 5, \dots, 6n + 1\}$ by

1.1 $f(u_iu_i')$ $=$ $2i-1$, for $i \in \{1,2,3,\dots,n\}$;

1.2 $f(u_iu)$ $=$ $2n+4i-3$, for $i \in \{1,2,3,\dots,n\}$;

1.3 $f(u_i'u')$ $=$ $2n+4i-1$, for $i \in \{1,2,3,\dots,n\}$;

1.4 $f(uu')$ $=$ $6n+1$.

Example 6.2 From Algorithm 6.1, we can label each edge of $\mathbf{Prism}(S_6)$ as shown in Figure 6.2.

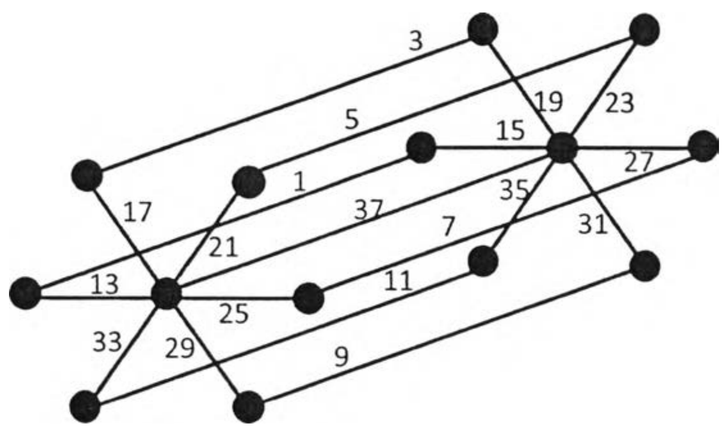

Figure 6.2 Edge-labeling for $\mathrm{Prism}(S_6)$.

Algorithm 6.2

Let $n > 3$ be an integer of the form $n = 6k-1$ for some $k \in \mathbb{N}$. Let G denote $\mathrm{Prism}(S_n)$. Then, $q = 3n+1$. Define $f\colon E(G) \to \{1,3,5,\dots,6n+1\}$ by

2.1 $f(u_iu_i')$ $=$ $2n+2i-1$, for $i \in \{1,2,3,\dots,n\}$;

2.2 $f(u_nu)$ $=$ 1;

2.3 $f(u_iu)$ $=$ $2i+1$, for $i \in \{1,2,3,\dots,n-1\}$;

2.4 $f(u_i'u')$ $=$ $4n+2i-1$, for $i \in \{1,2,3,\dots,n\}$;

2.5 $f(uu')$ $=$ $6n + 1$.

Example 6.3 From Algorithm 6.2, since $5 = 6(1) - 1$, we can label each edge of $\mathrm{Prism}(S_5)$ as shown in Figure 6.3.

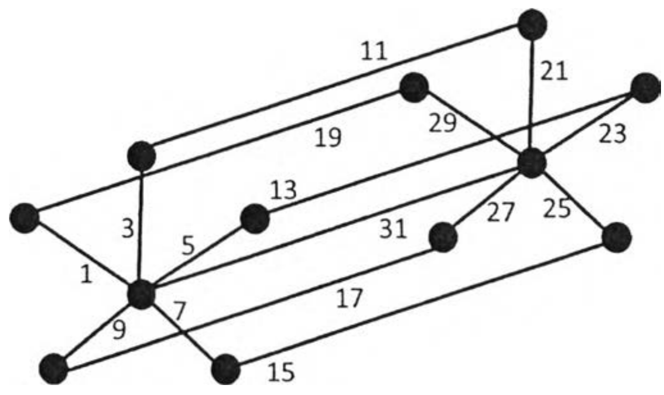

Figure 6.3 Edge-labeling for $\mathrm{Prism}(S_5)$.

Algorithm 6.3

Let $n > 3$ be an integer of the form $n = 6k + 1$ for some $k \in \mathbb{N}$. Let G denote $\mathrm{Prism}(S_n)$. Then, $q = 3n + 1$. Define $f: E(G) \to \{1, 3, 5, \dots, 6n + 1\}$ by

3.1 $f(u_i u_i')$ $=$ $4n + 2i + 1$, for $i \in \{1, 2, 3, \dots, n\}$;

3.2 $f(u_i u)$ $=$ $2i + 1$, for $i \in \{1, 2, 3, \dots, n\}$;

3.3 $f(u_1' u')$ $=$ 1;

3.4 $f(u_i' u')$ $=$ $2n + 2i + 1$, for $i \in \{2, 3, 4, \dots, n\}$;

3.5 $f(uu')$ $=$ $2n + 3$.

Example 6.4 From Algorithm 6.3, since $7 = 6(1) + 1$, we can label each edge of $\text{Prism}(S_7)$ as shown in Figure 6.4.

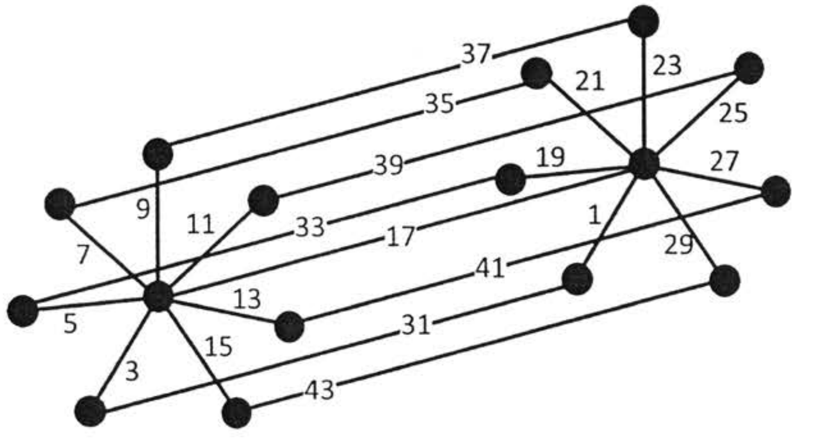

Figure 6.4 Edge-labeling for $\text{Prism}(S_7)$.

Algorithm 6.4

Let $n > 3$ be an integer of the form $n = 6k + 3$ for some $k \in \mathbb{N}$. Let G denote $\text{Prism}(S_n)$. Then, $q = 3n + 1$. Define $f: E(G) \rightarrow \{1, 3, 5, \ldots, 6n + 1\}$ by

4.1 $f(u_i u_i') = 4n + 2i + 1$, for $i \in \{1, 2, 3, \ldots, n\}$;

4.2 $f(u_i u) = 2i + 1$, for $i \in \{1, 2, 3, \ldots, n\}$;

4.3 $f(u_i' u') = 2n + 2i + 1$, for $i \in \{1, 2, 3, \ldots, n\}$;

4.4 $f(uu') = 1$.

Example 6.5 From Algorithm 6.4, since $9 = 6(1) + 3$, we can label each edge of $\text{Prism}(S_9)$ as shown in Figure 6.5.

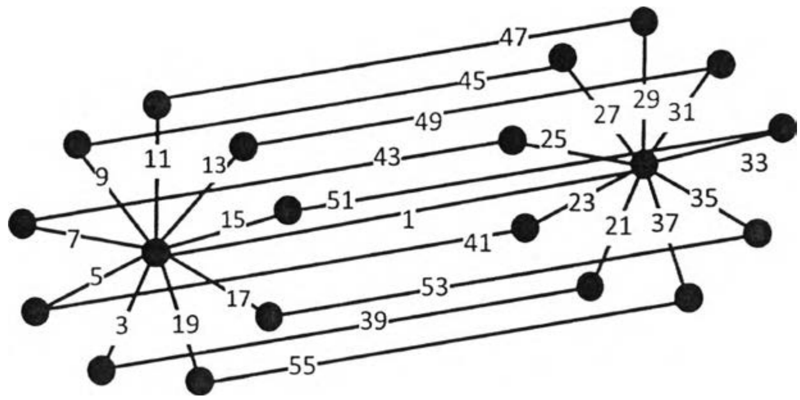

Figure 6.5 Edge-labeling for $\mathrm{Prism}(S_9)$.

Next, we show that if $n \geq 3$ and n is even, then $\mathrm{Prism}(S_n)$ is an edge-odd graceful graph.

Lemma 6.1 *Let $n \geq 3$ be even. $Prism(S_n)$ is an edge-odd graceful graph.*

Proof Let $n \geq 3$ be an even integer. We first prove that the function f defined in Algorithm 6.1 is a bijection from $E(G)$ to $\{1, 3, 5, \dots, 6n + 1\}$.

From Algorithm 6.1(1.1), we have

$$A = \{f(u_i u_i') \mid i \in \{1, 2, 3, \dots, n\}\} = \{1, 3, 5, \dots, 2n - 1\}.$$

From Algorithm 6.1(1.2), we have

$$B = \{f(u_i u) \mid i \in \{1, 2, 3, \dots, n\}\}$$

$$= \{2n + 1, 2n + 5, 2n + 9, \dots, 6n - 3\}.$$

From Algorithm 6.1(1.3), we have

$$C = \{f(u_i u') \mid i \in \{1, 2, 3, \dots, n\}\}$$

$$= \{2n + 3, 2n + 7, 2n + 11, \dots, 6n - 1\}.$$

From Algorithm 6.1(1.4), we have

$$D = \{f(uu')\} = \{6n+1\}.$$

We can see clearly that A, B, C and D are disjoint and

$$f\left(E\left(\text{Prism }(S_n)\right)\right) = A \cup B \cup C \cup D = \{1, 3, 5, \dots, 6n+1\}.$$

Next, we will show that the induced vertex-labels from the edge-labels using Algorithm 6.1 are in $\{0, 1, 2, \dots, 6n+1\}$ and all distinct. From Algorithm 6.1, we have

$$\begin{aligned} f^+(u_i) &= \left(f(u_iu_i') + f(u_iu)\right) \pmod{6n+2} \\ &= \left((2i-1) + (2n+4i-3)\right) \pmod{6n+2} \\ &= (2n+6i-4) \pmod{6n+2}, \text{ for } i \in \{1, 2, 3, \dots, n\}. \end{aligned}$$

$$\begin{aligned} f^+(u_i') &= \left(f(u_iu_i') + f(u_i'u')\right) \pmod{6n+2} \\ &= \left((2i-1) + (2n+4i-1)\right) \pmod{6n+2} \\ &= (2n+6i-2) \pmod{6n+2}, \text{ for } i \in \{1, 2, 3, \dots, n\}. \end{aligned}$$

$$\begin{aligned} f^+(u) &= \left(\textstyle\sum_{i=1}^{n} f(u_iu) + f(uu')\right) \pmod{6n+2} \\ &= \left(\textstyle\sum_{i=1}^{n} (2n+4i-3) + 6n+1\right) \pmod{6n+2} \\ &= \left((2n^2 + (2n^2+2n) - 3n) + 6n+1\right) \pmod{6n+2} \\ &= (4n^2+5n+1) \pmod{6n+2}. \end{aligned}$$

$$\begin{aligned} f^+(u') &= \left(\textstyle\sum_{i=1}^{n} f(u_i'u') + f(uu')\right) \pmod{6n+2} \\ &= \left(\textstyle\sum_{i=1}^{n} (2n+4i-1) + 6n+1\right) \pmod{6n+2} \\ &= \left((2n^2 + (2n^2+2n) - n) + 6n+1\right) \pmod{6n+2} \\ &= (4n^2+7n+1) \pmod{6n+2}. \end{aligned}$$

Next, we will show that $f^+(u_i)$, $f^+(u_i')$, $f^+(u)$ and $f^+(u')$ are distinct by using the contradiction argument. Let $i, j \in \{1,2,3,\dots,n\}$ and $i \neq j$, we first suppose that $f^+(u_i) \equiv f^+(u_j) \pmod{6n+2}$ and $f^+(u_i') \equiv f^+(u_j') \pmod{6n+2}$.

Since $f^+(u_i) \equiv f^+(u_j) \pmod{6n+2}$,

$$2n + 6i - 4 \equiv 2n + 6j - 4 \pmod{6n+2}.$$

This implies that $6(i-j) \equiv 0 \pmod{6n+2}$, which is a contradiction.

Since $f^+(u_i') \equiv f^+(u_j') \pmod{6n+2}$,

$$2n + 6i - 2 \equiv 2n + 6j - 2 \pmod{6n+2}.$$

This implies that $6(i-j) \equiv 0 \pmod{6n+2}$, which is a contradiction.

Second, let $i \in \{1,2,3,\dots,n\}$ and suppose that $f^+(u_i) \equiv f^+(u_i') \pmod{6n+2}$. Then,

$$2n + 6i - 4 \equiv 2n + 6i - 2 \pmod{6n+2}.$$

This implies that $2 \equiv 0 \pmod{6\text{n}+2}$, which is a contradiction.

Next, let $i, j \in \{1,2,3,\dots,n\}$ and $i \neq j$ suppose that $f^+(u_i) \equiv f^+(u_j') \pmod{6n+2}$. Then, $2n + 6i - 4 \equiv 2n + 6j - 2 \pmod{6n+2}$. This implies that

$$3(i-j) \equiv 1 \pmod{3\text{n}+1}.$$

Since $3(2n+1) \equiv 1 \pmod{3\text{n}+1}$ and for $i, j \in \{1,2,3,\dots,n\}$, $(i-j) \pmod{3\text{n}+2} \in \{1,2,3,\dots,n-1,2n+2,2n+3,2n+4,\dots,3n\}$, we get a contradiction.

Finally, suppose that $f^+(u) \equiv f^+(u') \pmod{6n+2}$. Then,

$$4n^2 + 5n + 1 \equiv 4n^2 + 7n + 1 \pmod{6n+2}.$$

This implies that $2n \equiv 0 \pmod{6n+2}$, which is a contradiction.

Thus, $f^+(u_i), f^+(u_i'), f^+(u)$ and $f^+(u')$ are all distinct and all are subsets of $\{0, 1, 2, \ldots, 6n + 1\}$. Therefore, the function f defined in Algorithm 6.1 is an edge-odd graceful labeling. ■

Lemma 6.2 *$Prism(S_3)$ is an edge odd graceful labeling.*

Proof. we can label edge by Figure 6.1.

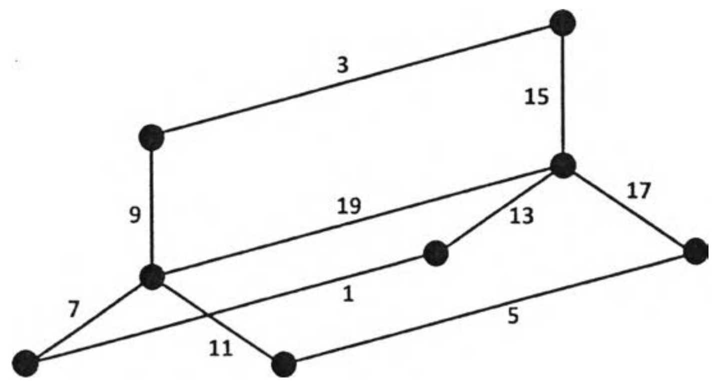

Figure 6.6 Edge-labeling for $\mathrm{Prism}(S_3)$.

The vertices of $\mathrm{Prism}(S_3)$ induced by edge labeling shown in Figure 6.7

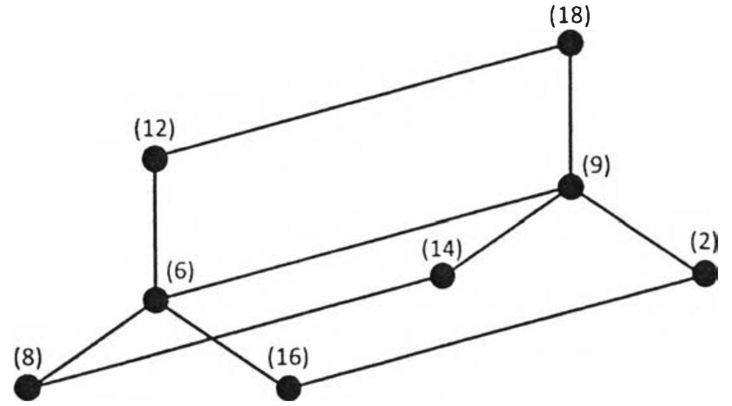

Figure 6.7 The vertex-labeling for $\mathrm{Prism}(S_3)$ induced by Figure 6.6.

Thus, $\mathrm{Prism}(S_3)$ is an edge-odd graceful graph. ■

Lemma 6.3 *Let* $n \geq 3$. *If* $n = 6k - 1$ *for some* $k \in \mathbb{N}$, *then* $4|a$ and $4|b$, where $a = n^2 - 1 \pmod{6n+2}$ and $b = 5n^2 - 1 \pmod{6n+2}$.

Proof. *Let* $n \geq 3$ and there is $k \in \mathbb{N}$ such that $n = 6k - 1$. Then,

$$a = n^2 - 1 = (6k-1)^2 - 1 = 36k^2 - 12k$$

$$\equiv -8k \pmod{36k - 4}$$

$$\equiv 28k - 4 \pmod{36k - 4}$$

$$\equiv 4(7k - 1) \pmod{36k - 4}$$

and

$$b = 5n^2 - 1 = 5(6k-1)^2 - 1 = 180k^2 - 60k + 4$$

$$\equiv -40k + 4 \pmod{36k - 4}$$

$$\equiv 32k - 4 \pmod{36k - 4}$$

$$\equiv 4(8k + 1) \pmod{36k - 4}.$$

That is, $4|a$ and $4|b$, where $a = n^2 - 1 \pmod{6n+2}$ and $b = 5n^2 - 1 (\bmod\ 6n + 2)$. ■

Lemma 6.4 *Let* $n \geq 3$. *If* $n = 6k - 1$ *for some* $k \in \mathbb{N}$, *then* $Prism(S_n)$ *is an edge-odd graceful graph.*

Proof. Let $n \geq 3$ and there is $k \in \mathbb{N}$ such that $n = 6k - 1$. We first prove that the function f defined in Algorithm 6.2 is a bijection from $E(G)$ to $\{1, 3, 5, \ldots, 6n+1\}$.

From Algorithm 6.2(2.1), we have

$$A = \{f(u_i u_i') \mid i \in \{1, 2, 3, \ldots, n\}\}$$

$$= \{2n+1, 2n+3, 2n+5, \ldots, 4n-1\}.$$

From Algorithm 6.2(2.2 and 2.3), we have

$$B = \{f(u_i u) \mid i \in \{1, 2, 3, \ldots, n-1\}\} \cup \{f(u_n u)\}$$

$$= \{3, 5, 7, \dots, 2n-1\} \cup \{1\}.$$

From Algorithm 6.2(2.4), we have

$$C \quad = \{f(u_i u') \mid i \in \{1, 2, 3, \dots, n\}\}$$

$$= \{4n+1, 4n+3, 4n+5, \dots, 6n-1\}.$$

From Algorithm 6.2(2.5), we have

$$D \quad = \{f(uu')\} = \{6n+1\}.$$

We can see clearly that A, B, C and D are disjoint and

$$f\left(E\left(\text{Prism}(S_n)\right)\right) = A \cup B \cup C \cup D = \{1, 3, 5, \dots, 6n+1\}.$$

Next, we will show that the induced vertex-labels from the edge-labels using Algorithm 6.2 are in $\{0, 1, 2, \dots, 6n+1\}$ and all distinct. From Algorithm 6.2, we have

$$f^+(u_i) \quad = \left(f(u_i u_i') + f(u_i u)\right) \pmod{6n+2}$$

$$= \left((2n+2i-1) + (2i+1)\right) \pmod{6n+2}$$

$$= 2n + 4i, \text{ for } i \in \{1, 2, 3, \dots, n-1\};$$

$$f^+(u_n) \quad = \left(f(u_n u'_n) + f(u'_n u')\right) \pmod{6n+2}$$

$$= \left((2n+2n-1) + 1\right) \pmod{6n+2}$$

$$= 4n;$$

$$f^+(u_i') \quad = \left(f(u_i u_i') + f(u_i' u')\right) \pmod{6n+2}$$

$$= \left((2n+2i-1) + (4n+2i-1)\right) \pmod{6n+2}$$

$$= (6n+4i-2) \pmod{6n+2}$$

$$= 4i - 4, \text{ for } i \in \{1, 2, 3, \dots, n\};$$

$$
\begin{aligned}
f^+(u) &= \left(\textstyle\sum_{i=1}^{n} f(u_iu) + f(uu')\right) \pmod{6n+2} \\
&= \left(\textstyle\sum_{i=1}^{n}(2i-1) + (6n+1)\right) \pmod{6n+2} \\
&= \left(n^2 + (6n+1)\right) \pmod{6n+2} \\
&= (n^2 - 1) \pmod{6n+2};
\end{aligned}
$$

$$
\begin{aligned}
f^+(u') &= \left(\textstyle\sum_{i=1}^{n} f(u_i'u') + f(uu')\right) \pmod{6n+2} \\
&= \left(\textstyle\sum_{i=1}^{n}(4n+2i-1) + (6n+1)\right) \pmod{6n+2} \\
&= \left((4n^2 + (n^2+n) - n) + (6n+1)\right) \pmod{6n+2} \\
&= (5n^2 + 6n + 1) \pmod{6n+2} \\
&= (5n^2 - 1) \pmod{6n+2}.
\end{aligned}
$$

Next, we will show that $f^+(u_i)$, $f^+(u_i')$, $f^+(u)$ and $f^+(u')$ are distinct. Since

$$
\begin{aligned}
&\{f^+(u_i) \mid i \in \{1,2,3,\dots,n-1\}\} \cup \{f^+(u_n)\} \\
&\qquad = \{2n+4, 2n+8, 2n+12, \dots, 6n-4\} \cup \{4n\}
\end{aligned}
$$

and

$$\{f^+(u_i') \mid i \in \{1,2,3,\dots,n\}\} = \{0, 4, 8, \dots, 4n-4\},$$

$\{f^+(u_i) \mid i \in \{1,2,3,\dots,n\}\}$ and $\{f^+(u_i') \mid i \in \{1,2,3,\dots,n\}\}$ are disjoint. By Lemma 6.3, we have $4 \mid f^+(u)$ and $4 \mid f^+(u')$. Then, if we need $f^+(u_i)$, $f^+(u_i')$, $f^+(u)$ and $f^+(u')$ to be distinct, we must show that the values of $f^+(u)$ and $f^+(u')$ under the integers modulo $6n+2$ are greater than $4n \pmod{6n+2}$. Since $n = 6k-1$, $6n+2 = 36k-4$, $n^2-1 = 36k^2-12k$ and $5n^2-1 = 180k^2-60k+4$. Then,

$$
\begin{aligned}
f^+(u) &= (n^2 - 1) \pmod{6n+2} \\
&= (36k^2 - 12k) \pmod{36k-4}
\end{aligned}
$$

$$\begin{aligned}
&\equiv -8k \pmod{36k-4} \\
&\equiv (28k-4) \pmod{36k-4} \\
&> (24k-4) \pmod{36k-4} \\
&= 4n \pmod{6n+2},
\end{aligned}$$

and

$$\begin{aligned}
f^+(u') &= (5n^2-1) \pmod{6n+2} \\
&= (180k^2-60k+4) \pmod{36k-4} \\
&\equiv (-40k+4) \pmod{36k-4} \\
&\equiv -4k \pmod{36k-4} \\
&\equiv (32k-4) \pmod{36k-4} \\
&> (24k-4) \pmod{36k-4} \\
&\equiv 4n \pmod{6n+2}.
\end{aligned}$$

Hence, $f^+(u_i)$, $f^+(u_i')$, $f^+(u)$ and $f^+(u')$ are distinct and they are subsets of $\{0,1,2,\dots,6n+1\}$. Therefore, the function f defined in Algorithm 6.2 is an edge-odd graceful labeling for each $n = 6k-1$ with $k \in \mathbb{Z}$. ■

Lemma 6.5 *Let $n \geq 3$. If $n = 6k+1$ for some $k \in \mathbb{N}$, then $Prism(S_n)$ is an edge-odd graceful graph.*

Proof. Let $n \geq 3$. Assume that there is $k \in \mathbb{N}$ such that $n = 6k+1$. We first prove that the function f defined in Algorithm 6.3 is a bijection from $E(G)$ to $\{1,3,5,\dots,6n+1\}$.

From Algorithm 6.3(3.1), we have

$$\begin{aligned}
A &= \{f(u_iu_i') \mid i \in \{1,2,3,\dots,n\}\} \\
&= \{4n+3, 4n+5, 4n+7, \dots, 6n+1\}.
\end{aligned}$$

From Algorithm 6.3(3.2), we have

$$B = \{f(u_iu) \mid i \in \{1,2,3,\dots,n\}\} = \{3,5,7,\dots,2n+1\}.$$

From Algorithm 6.3(3.3 and 3.4), we have

$$C = \{f(u_1'u')\} \cup \{f(u_i'u') \mid i \in \{2,3,4,\dots,n\}\}$$

$$= \{1\} \cup \{2n+5, 2n+7, 2n+9, \dots, 4n+1\}.$$

From Algorithm 6.3(3.5), we have

$$D = \{f(uu')\} = \{2n+3\}.$$

We can see clearly that A, B, C and D are disjoint and

$$f\left(E(\mathrm{Prism}(S_n))\right) = A \cup B \cup C \cup D = \{1,3,5,\dots,6n+1\}.$$

Next, we will show that the induced vertex-labels from the edge-labels using Algorithm 6.3 are in $\{0,1,2,\dots,6n+1\}$ and all distinct. From Algorithm 6.3, we have

$$f^+(u_i) = \left(f(u_iu_i') + f(u_iu)\right) \pmod{6n+2}$$

$$= \left((4n+2i+1) + (2i+1)\right) \pmod{6n+2}$$

$$= (4n+4i+2) \pmod{6n+2}, \text{ for } i \in \{1,2,3,\dots,n\};$$

$$f^+(u) = \left(\textstyle\sum_{i=1}^{n} f(u_iu) + f(uu')\right) \pmod{6n+2}$$

$$= \left(\textstyle\sum_{i=1}^{n}(2i+1) + 2n+3\right) \pmod{6n+2}$$

$$= \left((n^2+n+n) + 2n+3\right) \pmod{6n+2}$$

$$= (n^2+4n+3) \pmod{6n+2};$$

$$f^+(u_1') = \left(f(u_1u_1') + f(u_1'u')\right) \pmod{6n+2}$$

$$= \left((4n+3)+1\right) \pmod{6n+2}$$

$$= 4n + 4;$$

$$\begin{aligned} f^+(u_i') &= \big(f(u_iu_i') + f(u_i'u')\big) \pmod{6n+2} \\ &= \big((4n+2i+1) + (2n+2i+1)\big) \pmod{6n+2} \\ &= (6n+4i+2) \pmod{6n+2} \\ &= 4i, \text{ for } i \in \{2,3,4,\dots,n\}; \end{aligned}$$

$$\begin{aligned} f^+(u') &= \big(\textstyle\sum_{i=2}^{n} f(u_i'u') + f(u_1'u') + f(uu')\big) \pmod{6n+2} \\ &= \big(\textstyle\sum_{i=2}^{n}(2n+2i+1) + 1 + (2n+3)\big) \pmod{6n+2} \\ &= \big((2n^2 + (n^2+2n) - 2n - 3) + 1 + 2n + 3\big) \pmod{6n+2} \\ &= (3n^2 + 2n + 1) \pmod{6n+2}. \end{aligned}$$

Next, we will show that $f^+(u_i)$, $f^+(u_i')$, $f^+(u)$ and $f^+(u')$ are distinct.

Since $f^+(u_i) = 4n + 4i + 2 \pmod{6\text{n}+2}$ for $i \in \{1,2,3,\dots,n\}$ and $n = 6k+1$, $\{f^+(u_i) \mid i \in \{1,2,3,\dots,n\}\}$ can be divided into two sets as follows.

$$\left\{f^+(u_i) \,\middle|\, i \in \left\{1,2,3,\dots,\frac{n-1}{2}\right\}\right\} \cup \left\{f^+(u_i) \,\middle|\, i \in \left\{\frac{n+1}{2},\frac{n+3}{2},\frac{n+5}{2},\dots,n\right\}\right\}$$

$$= \{4n+6, 4n+10, 4n+14, \dots, 6n\} \cup \{2,6,10,\dots,2n\}$$

$$= \{24k+10, 24k+14, 24k+18, \dots, 36k+6\} \cup \{2,6,10,\dots,12k+2\}$$

and we have

$$\begin{aligned} \{f^+(u_i') \mid i \in \{1,2,3,\dots,n\}\} &= \{8,12,16,\dots,4n,4n+4\} \\ &= \{8,12,16,\dots,24k+4,24k+8\}. \end{aligned}$$

Since $n = 6k+1$, $6n+2 = 36k+8$, $n^2+4n+3 = 36k^2+36k+8$ and $3n^2 + 2n + 1 = 108k^2 + 48k + 6$. Then,

$$f^+(u) = (n^2+4n+3) \pmod{6n+2}$$

$$= (36k^2 + 36k + 8) \pmod{36k + 8}$$

$$= 28k + 8.$$

and

$$f^+(u') = (3n^2 + 2n + 1) \pmod{6n + 2}$$

$$= (180k^2 + 48k + 6) \pmod{36k + 8}$$

$$= 24k + 6.$$

Hence, $f^+(u_i)$, $f^+(u_i')$, $f^+(u)$ and $f^+(u')$ are distinct and they are subsets of $\{0, 1, 2, \dots, 6n + 1\}$. Therefore, the function f defined in Algorithm 6.3 is an edge-odd graceful labeling for all $n = 6k + 1$ for all $k \in \mathbb{Z}$. ■

Lemma 6.6 *Let $n \geq 3$. If $n = 6k + 3$ for some $k \in \mathbb{N}$, then $Prism(S_n)$ is an edge-odd graceful graph.*

Proof. Let $n \geq 3$. Assume that there is $k \in \mathbb{N}$ such that $n = 6k + 3$. We first prove that the function f defined in Algorithm 6.4 is a bijection from $E(G)$ to $\{1, 3, 5, \dots, 6n + 1\}$.

From Algorithm 6.4(4.1), we have

$$A = \{f(u_i u_i') \mid i \in \{1, 2, 3, \dots, n\}\}$$

$$= \{4n + 3, 4n + 5, 4n + 7, \dots, 6n + 1\}.$$

From Algorithm 6.4(4.2), we have

$$B = \{f(u_i u) \mid i \in \{1, 2, 3, \dots, n\}\} = \{3, 5, 7, \dots, 2n + 1\}.$$

From Algorithm 6.4(4.3), we have

$$C = \{f(u_i' u') \mid i \in \{1, 2, 3, \dots, n\}\}$$

$$= \{2n + 3, 2n + 5, 2n + 7, \dots, 4n + 1\}.$$

From Algorithm 6.4(4.4), we have

$$D \quad = \{f(uu')\} = \{1\}.$$

We can see clearly that A, B, C and D are disjoint and

$$f\left(E\left(\text{Prism}(S_n)\right)\right) = A \cup B \cup C \cup D = \{1, 3, 5, \dots, 6n + 1\}.$$

Next, we will show that the induced vertex-labels from the edge-labels using Algorithm 6.4 are in $\{0, 1, 2, \dots, 6n + 1\}$ and disjoint. From Algorithm 6.4, we have

$$\begin{aligned} f^+(u_i) &= \left(f(u_iu_i') + f(u_iu)\right) \pmod{6n+2} \\ &= \left((4n + 2i + 1) + (2i + 1)\right) \pmod{6n+2} \\ &= (4n + 4i + 2) \pmod{6n+2}, \text{ for } i \in \{1, 2, 3, \dots, n\}; \end{aligned}$$

$$\begin{aligned} f^+(u) &= \left(\textstyle\sum_{i=1}^{n} f(u_iu) + f(uu')\right) \pmod{6n+2} \\ &= \left(\textstyle\sum_{i=1}^{n}(2i + 1) + 1\right) \pmod{6n+2} \\ &= \left((n^2 + n + n) + 1\right) \pmod{6n+2} \\ &= (n^2 + 2n + 1) \pmod{6n+2}; \end{aligned}$$

$$\begin{aligned} f^+(u_i') &= \left(f(u_iu_i') + f(u_i'u')\right) \pmod{6n+2} \\ &= \left((4n + 2i + 1) + (2n + 2i + 1)\right) \pmod{6n+2} \\ &= (6n + 4i + 2) \pmod{6n+2} \\ &= 4i, \text{ for } i \in \{1, 2, 3, \dots, n\}; \end{aligned}$$

$$\begin{aligned} f^+(u') &= \left(\textstyle\sum_{i=1}^{n} f(u_i'u') + f(uu')\right) \pmod{6n+2} \\ &= \left(\textstyle\sum_{i=1}^{n}(2n + 2i + 1) + 1\right) \pmod{6n+2} \\ &= \left((2n^2 + (n^2 + n) + n) + 1\right) \pmod{6n+2} \\ &= (3n^2 + 2n + 1) \pmod{6n+2}. \end{aligned}$$

Next, we will show that $f^+(u_i)$, $f^+(u_i')$, $f^+(u)$ and $f^+(u')$ are distinct.

Since $f^+(u_i) = 4n + 4i + 2 \pmod{6n+2}$ for $i \in \{1, 2, 3, \ldots, n\}$ and $n = 6k + 3$, $\{f^+(u_i) \mid i \in \{1, 2, 3, \ldots, n\}\}$ can be divided into two sets as follows.

$$\left\{f^+(u_i) \middle| i \in \left\{1, 2, 3, \ldots, \frac{n-1}{2}\right\}\right\} \cup \left\{f^+(u_i) \middle| i \in \left\{\frac{n+1}{2}, \frac{n+3}{2}, \frac{n+5}{2}, \ldots, n\right\}\right\}$$

$$= \{4n + 6, 4n + 10, 4n + 14, \ldots, 6n\} \cup \{2, 6, 10, \ldots, 2n\}$$

$$= \{24k + 18, 24k + 22, 24k + 26, \ldots, 36k + 18\} \cup \{2, 6, 10, \ldots, 12k + 6\}$$

and

$$\{f^+(u_i') \mid i \in \{1, 2, 3, \ldots, n\}\} = \{4, 8, 12, \ldots, 4n\} = \{4, 8, 12, \ldots, 24k + 12\}.$$

Since $n = 6k + 3$, $6n + 2 = 36k + 20$, $n^2 + 2n + 1 = 36k^2 + 48k + 16$ and $3n^2 + 2n + 1 = 108k^2 + 120k + 34$. Then, we have

$$\begin{aligned} f^+(u) &= (n^2 + 2n + 1) \pmod{6n + 2} \\ &= (36k^2 + 48k + 16) \pmod{36k + 20} \\ &= 28k + 16 \end{aligned}$$

and

$$\begin{aligned} f^+(u') &= (3n^2 + 2n + 1) \pmod{6n + 2} \\ &= (108k^2 + 120k + 34) \pmod{36k + 20} \\ &= 24k + 14\,. \end{aligned}$$

Hence, $f^+(u_i)$, $f^+(u_i')$, $f^+(u)$ and $f^+(u')$ are distinct and they are subsets of $\{0, 1, 2, \ldots, 6n + 1\}$. Therefore, the function f defined in Algorithm 6.4 is an edge-odd graceful labeling each $n = 6k + 3$ with $k \in \mathbb{Z}$. ■

From Lemmas 6.1, 6.2, 6.4, 6.5 and 6.6, we conclude our result as in the following theorem.

Theorem 6.1 *Let $n \geq 3$. $Prism(S_n)$ is an edge-odd graceful graph.*

Example 6.6 From the edge-labeling in Example 6.2, the induced vertex-labeling of $\mathrm{Prism}(S_6)$ is shown in Figure 6.8.

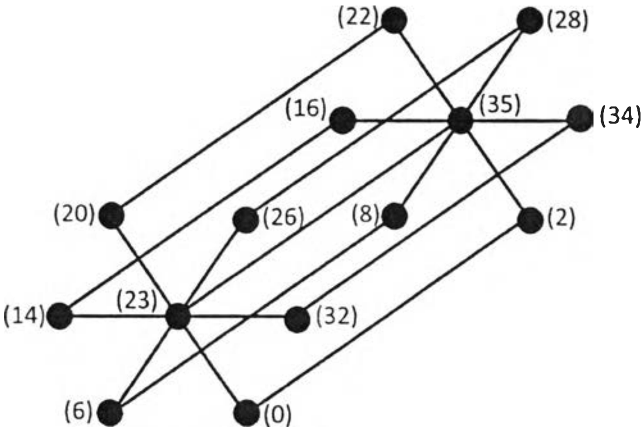

Figure 6.8 The vertex-labeling is induced from the edge-labeling in Figure 6.2.

Example 6.7 From the edge-labeling in Example 6.3, the induced vertex-labeling of $\mathrm{Prism}(S_5)$ is shown in Figure 6.9.

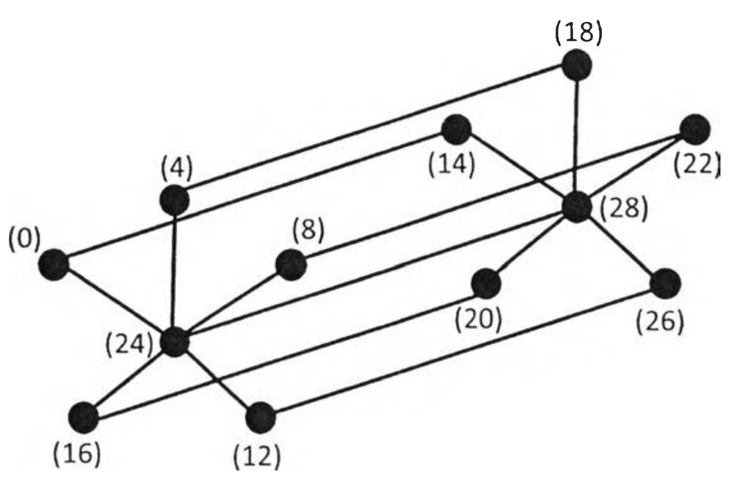

Figure 6.9 The vertex-labeling is induced from the edge-labeling in Figure 6.3.

Example 6.8 From the edge-labeling in Example 6.4, the induced vertex-labeling of $\mathrm{Prism}(S_7)$ is shown in Figure 6.10.

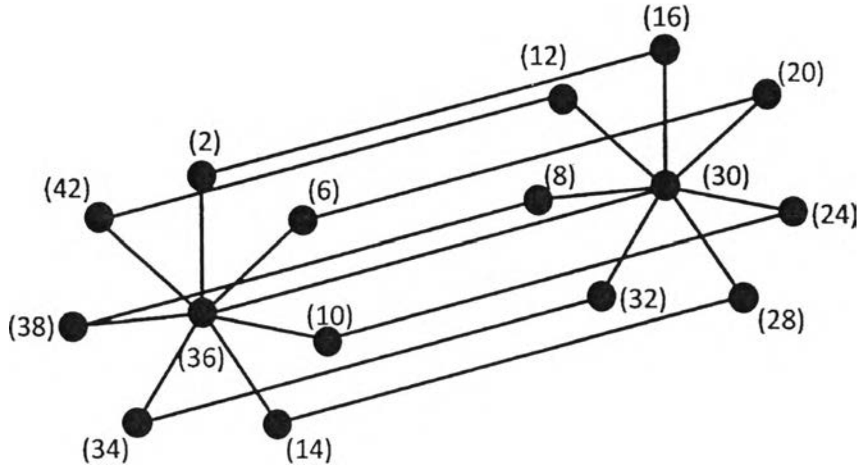

Figure 6.10 The vertex-labeling is induced from the edge-labeling in Figure 6.4.

Example 6.9 From the edge-labeling in Example 6.5, the induced vertex-labeling of $\mathrm{Prism}(S_9)$ is shown in Figure 6.11.

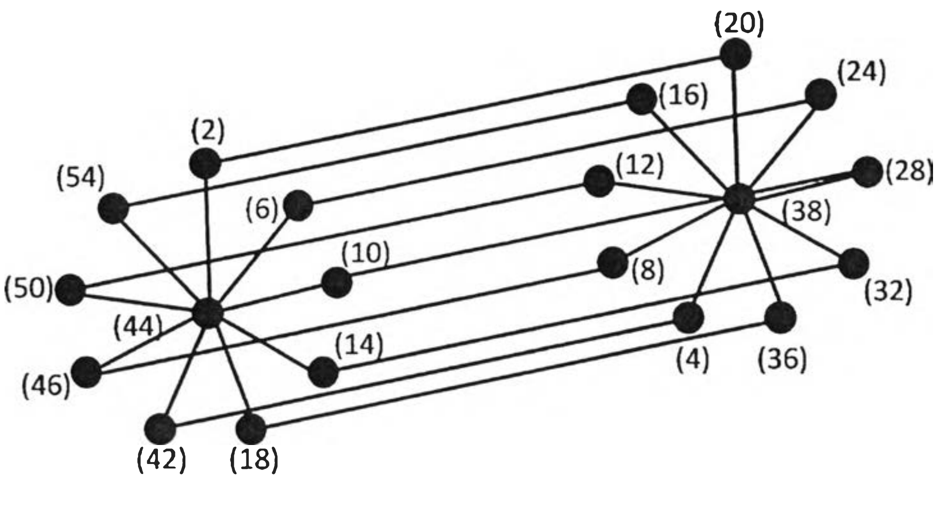

Figure 6.11 The vertex-labeling is induced from the edge-labeling in Figure 6.5.