CHAPTER II
PRELIMINARIES

Throughout, let $\mathbb{F}_q[x]$ be the ring of polynomials over the finite field $\mathbb{F}_q$ of characteristic p where q is a power of p, and let $\mathbb{F}_q(x)$ denote the quotient field of $\mathbb{F}_q[x]$. We begin with the definition of a valuation on an arbitrary field.

Definition 2.1. *[5] A **valuation** $|\cdot|$ on a field K is a real valued function with the following properties:*

1. *for all $\alpha \in K$, $|\alpha| \geq 0$ and $|\alpha| = 0$ if and only if $\alpha = 0$,*
2. *for all $\alpha, \beta \in K$, $|\alpha\beta| = |\alpha||\beta|$,*
3. *for all $\alpha, \beta \in K$, $|\alpha + \beta| \leq |\alpha| + |\beta|$.*

Definition 2.2. *[5] A valuation $|\cdot|$ on a field K is **non-archimedean** if the condition 3. in Definition 2.1 is replaced by a stronger condition, called the **strong triangle inequality***

$$|\alpha + \beta| \leq \max\{|\alpha|, |\beta|\}$$

*for all $\alpha, \beta \in K$. Any other valuation on K is called **archimedean**.*

Example 2.3.

1. The usual absolute value $|\cdot|$ is an archimedean valuation on $\mathbb{Q}$.
2. Define $|\cdot|$ on $\mathbb{F}_q(x)$ by $|0| = 0$ and for all $A, B \in \mathbb{F}_q[x]\backslash\{0\}$,

$$\left|\frac{A}{B}\right| = q^{\deg A - \deg B}.$$

Then $|\cdot|$ is a non-archimedean valuation on $\mathbb{F}_q(x)$ called the **degree valuation**.

Definition 2.4. *[5] A **function field** $\mathbb{F}_q((1/x))$ is a completion of $\mathbb{F}_q(x)$ with respect to the degree valuation $|\cdot|$, that is,*

$$\mathbb{F}_q\left(\left(\frac{1}{x}\right)\right) = \left\{a_k x^k + \cdots + a_1 x + a_0 + \frac{a_{-1}}{x} + \frac{a_{-2}}{x^2} + \cdots \mid a_i \in \mathbb{F}_q, k \in \mathbb{Z}\right\}.$$

2.1 The polynomials $\psi_k(t)$ and $G_k(t)$

In [1], Carlitz defined the polynomial $\psi_k(t)$ in $(\mathbb{F}_q[x])[t]$ which plays the role analogous to the binomial expansion and derived its property in the following theorem.

Definition 2.5. *[1] Define $\psi_0(t) = t$ and for $k \in \mathbb{N}$, define*

$$\psi_k(t) = \prod_{\deg M < k} (t - M),$$

where the product extends over all polynomials M (including 0) in an indeterminate x with coefficients in $\mathbb{F}_q$ of degree less than k.

Carlitz derived the formula of $\psi_k(t)$ as a polynomial in an indeterminate t whose coefficients are some certain polynomials.

Definition 2.6. *[1] Define $F_0 = 1$ and $L_0 = 1$. For $k \in \mathbb{N}$, define*

$$F_k = [k][k-1]^q[k-2]^{q^2} \cdots [1]^{q^{k-1}},$$
$$L_k = [k][k-1][k-2] \cdots [1],$$

where $[r] = x^{q^r} - x$ for all $r \in \mathbb{N}$.

Theorem 2.7. *[1] Let $k \in \mathbb{N}$. Then*

$$\psi_k(t) = \sum_{i=0}^{k} (-1)^{k-i} \begin{bmatrix} k \\ i \end{bmatrix} t^{q^i},$$

where

$$\begin{bmatrix} k \\ i \end{bmatrix} = \frac{F_k}{F_i L_{k-i}^{q^i}}.$$

As mentioned in [1],

$$\psi_k(x^k) = \psi_k(M) = F_k$$

for each monic polynomial M of degree k, F_k is the product of all monic polynomials in $\mathbb{F}_q[x]$ of degree k, and L_k is the least common multiple of all polynomials in $\mathbb{F}_q[x]$ of degree k.

In [2], Carlitz generalized $\psi_k(t)$ to the polynomial $G_k(t)$ and F_k to g_k.

Definition 2.8. *Define $d(0) := 0$ and for all $k \in \mathbb{N}$, if k is expressed with respect to the base q as*

$$k = \alpha_0 + \alpha_1 q + \alpha_2 q^2 + \cdots + \alpha_m q^m.$$

where $0 \leq \alpha_i < q$ and $\alpha_m \neq 0$, then we define $d(k) := m$. The number $d(k)$ is called the ***upper q-index*** *$d(k)$* ***of*** *k for all $k \in \mathbb{N}_0$.*

Definition 2.9. *[2] Define $G_0(t) = 1$ and $g_0 = 1$. For $k \in \mathbb{N}$, define*

$$G_k(t) = \psi_0^{\alpha_0}(t)\psi_1^{\alpha_1}(t)\cdots\psi_{d(k)}^{\alpha_{d(k)}}(t)$$

and

$$g_k = F_1^{\alpha_1}\cdots F_{d(k)}^{\alpha_{d(k)}}.$$

Remark 2.10.

1. For each $k \in \mathbb{N}$, we have $k < q^{d(k)+1}$ and $d(k) = \lfloor \log_q k \rfloor$.

2. For $0 \leq i < 2q$, we have $d(i) = 0$ or 1 and

$$i = \alpha_0 + \alpha_1 q.$$

   If $d(i) = 0$, then $\alpha_1 = 0$ and so

$$L_{d(i)} = 1 = F_0 = F_0^{\alpha_0} = g_i.$$

If $d(i) = 1$, then $\alpha_1 = 1$ and so

$$L_{d(i)} = [1] = F_1 = F_0^{\alpha_0} F_1^{\alpha_1} = g_i.$$

3. From the definition of $G_k(t)$, we have

$$G_{\alpha \cdot q^i}(t) = \psi_i^{\alpha}(t),$$

where $0 \leq \alpha < q$. Moreover, for $k \in \mathbb{N}$, we have

$$\deg G_k = \alpha_0 \cdot \deg \psi_0 + \alpha_1 \cdot \deg \psi_1 + \cdots + \alpha_{d(k)} \cdot \deg \psi_{d(k)} = k.$$

Definition 2.11. *[2] Define $G'_0(t) = 1$. For $k \in \mathbb{N}$, if k is expressed with respect to the base q as*

$$k = \alpha_0 + \alpha_1 q + \alpha_2 q^2 + \cdots + \alpha_{d(k)} q^{d(k)} \quad (0 \leqslant \alpha_i < q),$$

then we define

$$G'_k(t) = \prod_{i=0}^{d(i)} G'_{\alpha_i \cdot q^i}(t),$$

where

$$G'_{\alpha \cdot q^i}(t) = \begin{cases} \psi_i^{\alpha}(t) & \text{for } 0 \leq \alpha < q-1; \\ \psi_i^{\alpha}(t) - F_i^{\alpha} & \text{for } \alpha = q-1. \end{cases}$$

Theorem 2.12. *[2] Let $k \in \mathbb{N}_0$. For each $K \in \mathbb{F}_q[x]$, the expressions $\dfrac{G_k(K)}{g_k}$ and $\dfrac{G'_k(K)}{g_k}$ are in $\mathbb{F}_q[x]$.*

Theorem 2.13. *[2] Let $p(t)$ be a polynomial over $\mathbb{F}_q[x]$ in an indeterminate t of* *degree less than k. Then we have the unique representation*

$$p(t) = \sum_{i=0}^{k} A_i G_i(t).$$

Let $q^m > i$. Then the coefficient A_i is determined by

$$(-1)^m \frac{F_m}{L_m} A_i = \sum_{\deg K < m} G'_{q^m-1-i}(K) f(K).$$

Theorem 2.14. *[2] With the same notation as in Theorem 2.13, the coefficient A_i is also given by*

$$(-1)^m \frac{F_m}{L_m} A_i = \sum_{\substack{\deg K = m \\ K \text{ is monic}}} G'_{q^m-1-i}(K) f(K).$$

2.2 Integer-valued Differences for Polynomials

This section introduces the integral-valued differences for polynomials over $\mathbb{F}_q(x)$. In [7], Wagner defined the set $\bar{I}_r$ as follows.

Definition 2.15. *[7] Let $p(t) \in (\mathbb{F}_q(x))[t]$. For $M \in \mathbb{F}_q[x]\backslash\{0\}$, define the* ***difference of a polynomial*** *$p(t)$ as*

$$\Delta_M p(t) = \frac{p(t+M) - p(t)}{M}$$

and for nonzero elements $M_1, M_2, \ldots, M_r$ of $\mathbb{F}_q[x]$, let r^{th} ***difference of a polynomial*** *$p(t)$ be*

$$\Delta_{M_1, M_2, \ldots, M_r} p(t) = \Delta_{M_r}(\Delta_{M_1, M_2, \ldots, M_{r-1}} p(t)).$$

Definition 2.16. *[7] For any positive integer r, we define*

$$I_0 = \{p(t) \in (\mathbb{F}_q(x))[t] \mid p(d) \in \mathbb{F}_q[x] \text{ for all } d \in \mathbb{F}_q[x]\}.$$
$$I_r = \{p(t) \in (\mathbb{F}_q(x))[t] \mid \Delta_{m_1,m_2,\cdots,m_r} p(t) \in I_0 \text{ for } m_1, m_2, \ldots, m_r \in \mathbb{F}_q[x]\backslash\{0\}\}.$$
$$\bar{I}_r = I_0 \cap I_1 \cap \cdots \cap I_r.$$

Note that I_0 is the set of integral-valued polynomials. In [2], Carlitz derived the explicit shapes of elements in I_0 in the following theorem.

Theorem 2.17. *[2] Let $p(t) \in I_0$ of degree n. Then we may write*

$$p(t) = \sum_{i=0}^{n} A_i \frac{G_i(t)}{g_i},$$

where the A_i's are uniquely determined elements of $\mathbb{F}_q[x]$.

In [7], Wagner gave the necessary and sufficient conditions for polynomial $p(t)$ belonging to $\bar{I}_1$ and $\bar{I}_r$ for all $r \geq 1$ as stating in next two theorems.

Theorem 2.18. *[7] Let $p(t) \in I_0$ be as in the form in Theorem 2.17. Then $p(t) \in I_1$ if and only if, for all $i \geq 1$,*

$$L_{e^*(i)} \mid A_i,$$

where

$$e(i) = \max\{k \mid i \text{ is divisible by } q^k\},$$
$$e^*(i) = \max\{e(j) \mid 1 \leq j \leq i\}.$$

Theorem 2.19. *[7] Let $p(t)$ be given by Theorem 2.17. Then $p(t) \in I_r$ if and only if for all $i \geq r$,*

$$\frac{A_i}{L_{e(i_1)} L_{e(i_2)} \cdots L_{e(i_r)}} \in \mathbb{F}_q[x],$$

whenever $i_1, i_2, \ldots, i_r > 0$, $i_1 + i_2 + \cdots + i_r \leq i$ and the multinomial coefficient

$$\frac{i!}{i_1!i_2!\cdots i_r!(i-i_1-i_2-\cdots-i_r)!} \text{ is prime to } p.$$

Remark 2.20. For a non-negative integer i, we have $e^*(i)$ and the upper q index $d(i)$ of i are equal.