CHAPTER II
PRELIMINARIES

In this chapter, we provide elementary results that will be used in Chapter III and Chapter IV. First, we recall basic definitions and properties in module theory, for instance, direct summands and fully invariant submodules. Next, we focus on essential submodules, small submodules and projective modules.

2.1 Sums of Modules

We begin this section by giving elementary results of the sum of submodules, essential submodules and fully invariant submodules. Then we discuss a relationship between direct summands and idempotent elements. Moreover, we study properties of the intersections of direct summands and fully invariant submodules. Finally, we provide another form of the set of all endomorphisms of finite direct sum of modules.

In general, for any submodules N, K and L of M, $(N+K)\cap L \subseteq N+(K\cap L)$.

Proposition 2.1.1. $(Modular Law)$ *Let N, K and L be submodules of M such that $N \subseteq L$. Then $(N+K)\cap L = N+(K\cap L)$.*

Proof. It is clear that $(N+K)\cap L \subseteq N+(K\cap L)$ because $N \subseteq L$.

For the reverse of inclusion, let $m \in (N+K)\cap L$. Then $m = x+y$ for some $x \in N$ and $y \in K$. So $y = m-x \in L$, and then $y \in K\cap L$. Hence $m = x+y \in N+(K\cap L)$. Therefore, $(N+K)\cap L \subseteq N+(K\cap L)$. □

A submodule N of M is a *direct summand* of M, denoted by $N \leq^{\oplus} M$, if there is a submodule K of M such that $N+K = M$ and $N\cap K = 0$. We abbreviate this property as $N\oplus K = M$. Then every element $m \in M$ can be written uniquely as $m = x+y$ for some $x \in N$ and $y \in K$. An element e of R is an *idempotent* *element* of R if $e^2 = e$. Moreover, the phrase "$e^2 = e \in \operatorname{End}(M)$" means that e is an idempotent element of R.

Proposition 2.1.2. *Let N be a submodule of M. Then N is a direct summand of M if and only if $N = eM$ for some $e^2 = e \in \operatorname{End}(M)$. Moreover, if $e^2 = e \in \operatorname{End}(M)$, then $x = ex$ for any $x \in eM$.*

Proof. First, assume that $e^2 = e \in \operatorname{End}(M)$ and let $x \in eM$. Then $x = ey$ for some $y \in M$. Thus $x = ey = e^2(y) = e(e(y)) = ex$. Therefore, $x = ex$ for any $x \in eM$.

For the sufficiency, assume $e^2 = e \in \operatorname{End}(M)$. Then it is clear that $M = eM + (1-e)M$. Next, let $x \in eM \cap (1-e)M$. Then $x = ex$ and $x = (1-e)x = x - ex$. Thus $ex = x - ex$ so that $ex = e(x - ex) = ex - ex = 0$, i.e., $x = 0$. Therefore, $M = eM \oplus (1-e)M$. To show the necessity, assume that N is a direct summand of M. Then there is a submodule K of M such that $N \oplus K = M$. Define $e \in \operatorname{End}(M)$ by, for each $m \in M$, $e(m) = x$ where $m = x + y$ for some $x \in N$ and $y \in K$. Let $m \in M$. Then $e^2(m) = e(e(m)) = e(x) = e(x+0) = x = e(m)$. This shows that $e^2(m) = e(m)$ and $e(m) = x \in N$ for any $m \in M$ with $m = x + y$ where $x \in N$ and $y \in K$. Thus $e^2 = e$ and $eM \subseteq N$. Next, for $n \in N$, it follows that $n = e(n+0) \in e(M)$, i.e., $N \subseteq eM$. Therefore, $N = eM$. $\square$

Proposition 2.1.3. *Let e be an idempotent in* $\operatorname{End}(M)$*. Then $M = eM \oplus \ker e$ and $\ker e = (1-e)M$.*

Proof. Clearly, $e(1-e)M = 0$. So $(1-e)M \subseteq \ker e$. Next, let $x \in \ker e$. Then $ex = 0$. It follows that $x = (1-e)x \in (1-e)M$. Thus $\ker e \subseteq (1-e)M$. As a result, $\ker e = (1-e)M$. Therefore, $M = eM \oplus (1-e)M = eM \oplus \ker e$. $\square$

Proposition 2.1.4. *Let N and L be submodules of M. If $N \leq^{\oplus} M$ and $N \subseteq L$, then $N \leq^{\oplus} L$*

Proof. Assume that $N \leq^{\oplus} M$ and $N \subseteq L$. Then $M = N \oplus K$ for some submodule K of M. By Modular Law and $N \cap K = 0$, we obtain $L = M \cap L = (N \oplus K) \cap L = N \oplus (K \cap L)$. This forces that $N \leq^{\oplus} L$. $\square$

Proposition 2.1.5. *Let $M = N \oplus K$. Then the following statements holds.*

(i) There is $e^2 = e \in \mathrm{End}(M)$ such that $N = eM$ and $\ker e = K = (1-e)M$.

(ii) If $K \subseteq L$ for some submodule L of M, then there is $e^2 = e \in \mathrm{End}(M)$ such that $N = eM$ and $N \cap L = eL$.

Proof. (i) From the proof of Proposition 2.1.2, there is $e^2 = e \in \mathrm{End}(M)$ such that $N = eM$ where $e(m) = x$ with $m = x + y$, $x \in N$ and $y \in K$. So $K \subseteq \ker e$ because $e(y) = e(0 + y) = 0$ for all $y \in K$. Let $x + y \in \ker e$ where $x \in N$ and $y \in K$. Then $0 = e(x + y) = x$, it implies that $x + y = y \in K$. Thus $\ker e \subseteq K$ so that $K = \ker e$. Furthermore, $K = (1-e)M$ from Proposition 2.1.3.

(ii) Assume that $K \subseteq L$ for some submodule L of M. From (i), there is $e^2 = e \in \mathrm{End}(M)$ such that $N = eM$ and $\ker e = K = (1-e)M$. Since $K \subseteq L$, it follows that $L = \big(eM \oplus (1-e)M\big) \cap L = \big((eM \cap L) \oplus (1-e)M$. Let $x \in eM \cap L$. Then $x = ex = e(ex) \in eL$. So $N \cap L \subseteq eL$. Next, let $x \in eL$. Thus $x = ey$ for some $y \in L$ so that $y = u + v$ where $u = eu \in eM \cap L$ and $v = (1-e)v \in (1-e)M$. Hence $x = ey = e(u + v) = e\big(eu + (1-e)v\big) = eu \in eM \cap L$, this forces that $eL \subseteq eM \cap L$. Therefore, $N \cap L = eM \cap L = eL$. $\square$

Proposition 2.1.6. *Let N, K and L be submodules of M. If $M = N \oplus K = N \oplus L$, then $K \cong L$.*

Proof. Assume that $M = N \oplus K = N \oplus L$. From Proposition 2.1.5, there is $e^2 = e \in \mathrm{End}(M)$ such that $L = eM$ and $\ker e = N$. Then e is an epimorphism from $N \oplus L$ onto L. Notice that ei is a homomorphism from K to L where i is the inclusion homomorphism from K to M. Let $x, y \in K$ be such that $ei(x) = ei(y)$. Then $e(x) = e(y)$. So $x - y \in \ker e = N$. Thus $x - y \in N \cap K = 0$ implies that $x = y$. Hence ei is a monomorphism. Next, let $m \in L$. Then $m = em$ and $m = x + y$ for some $x \in N$ and $y \in K$. So $m = ex + ey = ey = (ei)(y)$ because $\ker e = N$. Hence ei is an epimorphism. Therefore, $K \cong L$. $\square$

The next corollary is an immediate result from Proposition 2.1.6.

Corollary 2.1.7. *Let N, K and L be submodules of M. If $M = N \oplus K = N \oplus L$ and $K \subseteq L$, then $K = L$.*

A submodule F of M is called a *fully invariant submodule* of M, denoted by $F \leq_{fully} M$, if $f(F) \subseteq F$ for all $f \in \text{End}(M)$.

Proposition 2.1.8. *Let N be a direct summand of M and F be a fully invariant submodule of M. Then*
(i) $N \cap F$ is a fully invariant submodule of N,
(ii) $N \cap F$ is a direct summand of F, and
(iii) $F = (N \cap F) \oplus (K \cap F)$ if $M = N \oplus K$.

Proof. (i) Let $N = eM$ for some $e^2 = e \in \text{End}(M)$. Then $eM \cap F \subseteq eF$ because $ex = e(ex) \in eF$ for any $ex \in eM \cap F$. Moreover, $eF \subseteq eM \cap F$ because $F \leq_{fully} M$. Thus $N \cap F = eM \cap F = eF$. Let $g \in \text{End}(N)$. Then $ge \in \text{End}(M)$. So $g(N \cap F) = g(eF) = ge(F) \subseteq F$, i.e., $g(N \cap F) \subseteq N \cap F$. Therefore, $N \cap F$ is a fully invariant submodule of N.

(ii) There is a submodule K of M such that $M = N \oplus K$. Then $N = eM$ and $K = fM$ where $e^2 = e, f^2 = f \in \text{End}(M)$. From the proof of (i), we obtain that $eF = N \cap F$ and $fF = K \cap F$. Then $F = eF \oplus fF = (N \cap F) \oplus (K \cap F)$. Thus $N \cap F$ is a direct summand of F.

(iii) This follows from the proof of (ii). □

Let $M = N \oplus K$. The homomorphism $\pi_N : M \to N$ defined by $\pi_N(x+y) = x$ for all $x \in N$ and $y \in K$ is an epimorphism and $(\pi_N)^2 = \pi_N$ and is called the *projection homomorphism from M onto N*. Moreover, the *projective homomorphism π_K from M onto K* can be defined similarly. Note that a submodule which is both a fully invariant submodule and a direct summand is called a *fully invariant direct summand.*

Proposition 2.1.9. *Let $M = N \oplus K$ and N be a fully invariant submodule of M. If K' is a fully invariant direct summand of K, then $N \oplus K'$ is a fully invariant direct summand of M.*

Proof. Assume that K' is a fully invariant submodule of K. Let $f \in \text{End}(M)$, $x \in N$ and $y \in K'$. Then $f(x) \in N$ because $N \leq_{fully} M$. Note that $f(y) = u + v$ for some $u \in N$ and $v \in K$. Observe that $u + v = \pi_N(u + v) + \pi_K(u + v)$. So

$f(y) = u+v = \pi_N(f(y))+\pi_K(f(y))$ and $\pi_K(f(y)) = \pi_K(f(i_K(y)))$ where i_K is the identity homomorphism on K. Moreover, $\pi_K f i_K(y) \in K'$ as $y \in K'$, $K' \leq_{fully} K$ and $\pi_K f i_K \in \mathrm{End}(K)$. Thus $f(x+y) = f(x)+f(y) = f(x)+\pi_N f(y)+\pi_K f i_K(y) \in N \oplus K'$. Therefore, $N \oplus K'$ is a fully invariant direct summand of M because K' is a direct summand of K and $M = N \oplus K$. $\square$

Let $M = N \oplus K$ and $g \in \mathrm{End}(N)$. Define $g \oplus 0_K : M \to M$ by $\big(g \oplus 0_K\big)(x+y) = g(x)+0_K(y)$, i.e., $\big(g \oplus 0_K\big)(x+y) = g(x)$ for all $x \in N$ and $y \in K$. Let $h \in \mathrm{End}(K)$. Then $0_N \oplus h$ can be defined similar to $g \oplus 0_K$ and $0_N \oplus h \in \mathrm{End}(M)$.

Lemma 2.1.10. *Let $M = N \oplus K$ and F be a fully invariant submodule of M. Let $g \in \mathrm{End}(N)$ and $h \in \mathrm{End}(K)$. Then the following statements hold.*

(i) $g^{-1}(N \cap F) = \big(g \oplus 0_K\big)^{-1}(F) \cap N$.

(ii) $g^{-1}(N \cap F) \oplus K = \big(g \oplus 0_K\big)^{-1}(F)$.

(iii) $h^{-1}(K \cap F) = \big(0_N \oplus h\big)^{-1}(F) \cap K$.

(iv) $N \oplus h^{-1}(K \cap F) = \big(0_N \oplus h\big)^{-1}(F)$.

Proof. (i) Let $x \in g^{-1}(N \cap F)$. Then $x \in N$ and $(g \oplus 0_K)(x) = g(x) \in N \cap F$. So $x \in \big(g \oplus 0_K\big)^{-1}(F) \cap N$. For the reverse of inclusion, let $x \in \big(g \oplus 0_K\big)^{-1}(F) \cap N$. Then $(g \oplus 0_K)(x) \in F$ and $x \in N$ so that $g(x) = (g \oplus 0_K)(x) \in N \cap F$. Thus $x \in g^{-1}(N \cap F)$. Therefore, $\big(g \oplus 0_K\big)^{-1}(F) \cap N = g^{-1}(N \cap F)$.

(ii) It is clear that $g^{-1}(N \cap F) \cap K = 0$ because $g^{-1}(N \cap F) \subseteq N$ and $N \cap K = 0$. Next, let $x + y \in g^{-1}(N \cap F) \oplus K$ where $x \in g^{-1}(N \cap F)$ and $y \in K$. Then $g(x) \in N \cap F$, so $\big(g \oplus 0_K\big)(x + y) = g(x) \in F$. Thus $x + y \in (g \oplus 0_K)^{-1}(F)$. For the reverse of inclusion, let $x + y \in (g \oplus 0_K)^{-1}(F)$ where $x \in N$ and $y \in K$. Then $g(x) \in N$ so that $g(x) = \big(g \oplus 0_K\big)(x + y) \in N \cap F$. Thus $x \in g^{-1}(N \cap F)$. Hence $x + y \in g^{-1}(N \cap F) \oplus K$. Therefore, $g^{-1}(N \cap F) \oplus K = \big(g \oplus 0_K\big)^{-1}(F)$.

The proofs of (iii) and (iv) can be shown similarly to ones of (i) and (ii), respectively. $\square$

Proposition 2.1.11. *Let $M = N \oplus K$ and $f \in \mathrm{End}(M)$. Then the following statements hold.*

(i) $f^{-1}(N) \cap K = \ker(\pi_K f|_K)$.

(ii) If N is a fully invariant submodule of M, then $f^{-1}(N) = N \oplus \ker(\pi_K f|_K)$.

Proof. (i) Since $\pi_K : M \to K$ is the projection homomorphism, $\ker \pi_K = N$. Let $x \in f^{-1}(N) \cap K$. Then $f(x) \in N$ and $x \in K$. So $\pi_K f|_K(x) = \pi_K f(x) = 0$. Thus $x \in \ker(\pi_K f|_K)$. On the other hand, let $x \in \ker(\pi_K f|_K)$. Then $x \in K$ and $\pi_K f(x) = \pi_K f|_K(x) = 0$ so that $f(x) \in \ker \pi_K = N$, i.e., $x \in f^{-1}(N)$. Thus $x \in f^{-1}(N) \cap K$.

(ii) Assume that N is a fully invariant submodule of M. Then $N \subseteq f^{-1}(N)$. Applying the Modular Law gives $N \oplus \ker(\pi_K f|_K) = N \oplus \big(f^{-1}(N) \cap K\big) = f^{-1}(N) \cap \big(N \oplus K\big) = f^{-1}(N) \cap M = f^{-1}(N)$. □

Let N be a submodule of M and $x \in M$. Recall that

$$(N :_R x) = \{a \in R \mid xa \in N\}.$$

Next, we consider the quotient submodule M/N. Then

$$\begin{aligned}\big(\{N\} :_R x + N\big) &= \big\{a \in R \mid (x+N)a \in \{N\}\big\}\\ &= \{a \in R \mid xa + N = N\}\\ &= \{a \in R \mid xa \in N\}\\ &= (N :_R x).\end{aligned}$$

Let I be a nonempty subset of $\operatorname{End}(M)$. Then

$$\begin{aligned}(N :_M I) &= \{x \in M \mid f(x) \in N \text{ for all } f \in I\}\\ &= \{x \in M \mid x \in f^{-1}(N) \text{ for all } f \in I\}\\ &= \bigcap_{f \in I} f^{-1}(N).\end{aligned}$$

In particular, if $f \in \operatorname{End}(M)$, then $(N :_M f) = (N :_M \{f\}) = \{x \in M \mid x \in f^{-1}(N)\} = f^{-1}(N)$. Denote Sf a left ideal of $S = \operatorname{End}(M)$ generated by f where $f \in \operatorname{End}(M)$.

Proposition 2.1.12. *Let $f, h \in \operatorname{End}(M)$ and F be a fully invariant submodule of M. Then the following statements hold.*

(i) $(F :_M f) = (F :_M Sf)$

(ii) $(F :_M Sf + Sh) = (F :_M Sf) \cap (F :_M Sh)$.

Proof. (i) Since F is a fully invariant submodule of M and $\mathrm{End}(M)$ has the identity, we obtain that

$$\begin{aligned}(F :_M Sf) &= \{x \in M \mid g(x) \in F \text{ for all } g \in Sf\} \\ &= \{x \in M \mid hf(x) \in F \text{ for all } h \in S\} \\ &= \{x \in M \mid f(x) \in F\} \\ &= (F :_M f).\end{aligned}$$

(ii) Let $x \in (F :_M Sf + Sh)$. Then $(g_1 f + g_2 h)(x) \in F$ for any $g_1, g_2 \in S$. In particular, $f(x) \in F$ and $h(x) \in F$ so that $x \in (F :_M f) = (F :_M Sf)$ and $x \in (F :_M h) = (F :_M Sh)$, respectively. On the other hand, let $x \in (F :_M Sf) \cap (F :_M Sh) = (F :_M f) \cap (F :_M h)$. Then $f(x) \in F$ and $h(x) \in F$. Thus, for any $g_1, g_2 \in S$, $(g_1 f + g_2 h)(x) = g_1 f(x) + g_2 h(x) \in F$ because $F \leq_{fully} M$. Hence $x \in (F :_M Sf + Sh)$. Therefore, $(F :_M Sf + Sh) = (F :_M Sf) \cap (F :_M Sh)$. $\square$

Proposition 2.1.13. [15] *Let M_1 and M_2 be R-modules. Then*

$$\mathrm{End}(M_1 \oplus M_2) \cong \begin{pmatrix} \mathrm{End}(M_1) & \mathrm{Hom}(M_2, M_1) \\ \mathrm{Hom}(M_1, M_2) & \mathrm{End}(M_2) \end{pmatrix}.$$

Moreover, any epimorphism in $\mathrm{End}(M_1 \oplus M_2)$ can be written as $\begin{pmatrix} f & g' \\ f' & g \end{pmatrix}$ where $f \in \mathrm{End}(M_1)$, $f' \in \mathrm{Hom}(M_1, M_2)$, $g' \in \mathrm{Hom}(M_2, M_1)$ and $g \in \mathrm{End}(M_2)$.

Proposition 2.1.14. [15] *Let M_i be an R-module for all $i \in \{1, \ldots, n\}$. Then*

$$\mathrm{End}(M_1 \oplus M_2 \oplus \cdots \oplus M_n) \cong \begin{pmatrix} \mathrm{End}(M_1) & \mathrm{Hom}(M_2, M_1) & \ldots & \mathrm{Hom}(M_n, M_1) \\ \mathrm{Hom}(M_1, M_2) & \mathrm{End}(M_2) & & \mathrm{Hom}(M_n, M_2) \\ \vdots & \vdots & & \vdots \\ \mathrm{Hom}(M_1, M_n) & \mathrm{Hom}(M_2, M_n) & \ldots & \mathrm{End}(M_n) \end{pmatrix}.$$

2.2 Essential Submodules

A submodule N of M is an *essential submodule* of M, denoted by $N \leq_{ess} M$, if $N \cap K \neq 0$ for any nonzero submodule K of M. Moreover, M is an *essential extension* of N if $N \leq_{ess} M$.

Proposition 2.2.1. [3] *Let N be a submodule of M. Then N is an essential submodule of M if and only if for any nonzero element $x \in M$, there is $r \in R$ such that $0 \neq xr \in N$.*

Proposition 2.2.2. *Let N be a submodule of M. If N is both a direct summand and an essential submodule of M, then $N = M$.*

Proof. Assume that N is a direct summand of M and an essential submodule of M. Then there is a submodule K of M such that $M = N \oplus K$. So $N \cap K = 0$. Thus $K = 0$ because $N \leq_{ess} M$. Therefore, $N = M$. $\square$

Proposition 2.2.3. [3] *Let N and L be submodules of M and $N \subseteq L$. Then $N \leq_{ess} M$ if and only if $N \leq_{ess} L$ and $L \leq_{ess} M$.*

Proposition 2.2.4. *Let N, K, L and P be submodules of M such that N and K are submodules of L and P, respectively. If $N \leq_{ess} L$ and $K \leq_{ess} P$, then $N \cap K \leq_{ess} L \cap P$.*

Proof. Assume that $N \leq_{ess} L$ and $K \leq_{ess} P$. Let A be a nonzero submodule of $L \cap P$. Then A is a nonzero submodule of both L and P. Since $K \leq_{ess} P$ and A is a nonzero submodule of P, it follows that $K \cap A \neq 0$. Since $N \leq_{ess} L$ and $K \cap A$ is a nonzero submodule of L, we obtain that $N \cap (K \cap A) \neq 0$ so that $(N \cap K) \cap A \neq 0$. Therefore, $N \cap K \leq_{ess} L \cap P$. $\square$

Corollary 2.2.5. [3] *Let N and K be submodules of M. Then $N \leq_{ess} M$ and $K \leq_{ess} M$ if and only if $N \cap K \leq_{ess} M$.*

Proposition 2.2.6. *Let P and M be modules and $f : P \to M$ be a homomorphism. For any submodules N and L of M, if $N \leq_{ess} L$, then $f^{-1}(N) \leq_{ess} f^{-1}(L)$.*

Proof. Let N and L be submodules of M. Assume that $N \leq_{ess} L$. Let $0 \neq x \in f^{-1}(L)$. If $f(x) = 0$, then $x \in f^{-1}(N)$. Assume that $0 \neq f(x) \in L$. Since $N \leq_{ess} L$, there is $r \in R$ such that $0 \neq f(xr) \in N$. So $0 \neq xr \in f^{-1}(N)$. Therefore, $f^{-1}(N) \leq_{ess} f^{-1}(L)$. $\square$

Proposition 2.2.7. [3] *Let P and M be modules. Let N be a submodule of M, L be a submodule of P and $M \cap P = 0$. Then $N \leq_{ess} M$ and $L \leq_{ess} P$ if and only if $N \oplus L \leq_{ess} M \oplus P$.*

2.3 Small Submodules

A submodule N of M is a *small submodule* of M, denoted by $N \ll M$, if $N + K = M$ implies $K = M$ for any submodule K of M.

Proposition 2.3.1. *Let N be a submodule of M. If N is both a direct summand and a small submodule of M, then $N = 0$.*

Proof. Assume that N is a direct summand of M and a small submodule of M. Then there is a submodule K of M such that $M = N \oplus K$. So $N + K = M$. Thus $K = M$ because $N \leq_{ess} M$. This forces that $N = 0$. □

Proposition 2.3.2. [7] *Let N and L be submodules of M and $N \subseteq L$. Then $L \ll M$ if and only if $N \ll M$ and $L/N \ll M/N$.*

Proposition 2.3.3. [7] *Let N, K and L be submodules of M. If $M = L + K$ and $N \subseteq L$, then $(L \cap N)/(L \cap K) = M/(L \cap K)$.*

Proposition 2.3.4. [7] *Let N be a submodule of M. Then $N \ll M$ if and only if $N \ll L$ for all direct summand L of M containing N.*

Proposition 2.3.5. [7] *Let N and L be submodules of M. Then $N \ll M$ and $L \ll M$ if and only if $N + L \ll M$.*

Proposition 2.3.6. [7] *Assume that $N \ll M$ and $f : M \rightarrow P$ is a homomorphism. Then $f(N) \ll P$.*

A submodule L of M *lies above a direct summand* of M, given by Clark et al. in [7], if there is a direct summand N of M such that $N \subseteq L$ and $L/N \ll M/N$. Observe that every direct summand of M always lies above itself; moreover, every small submodules of M always lies above the zero submodule. Next, we provide equivalent definitions of lying above a direct summand.

Proposition 2.3.7. [7] *Let L be a submodule of M. Then the following statements are equivalent.*

(i) L lies above a direct summand of M.

(ii) There is a direct summand N of M and a submodule K of M such that $N \subseteq L$, $L = N + K$ and $K \ll M$.

(iii) There is a decomposition $M = N \oplus K$ with $N \subseteq L$ and $K \cap L \ll K$.

(iv) $L = eM \oplus (1-e)L$ and $(1-e)L \ll M$ for some $e^2 = e \in \mathrm{End}(M)$.

2.4 Projective Modules

A module P is a *projective module* if for any modules M and Q any epimorphism $g : M \to Q$ and any homomorphism $f : P \to Q$, there is a homomorphism $h : P \to M$ such that $f = gh$ (see the following diagram).

$$\begin{tikzcd}
 & P \arrow[d, "f"] \arrow[dl, dotted, "h"'] & \\
M \arrow[r, "g"'] & Q \arrow[r] & 0
\end{tikzcd}$$

A module M is a *free module* if M is a module with basis. Moreover, every ring is both a free module and a projective module over itself. Note that all of propositions in this section are from [15].

Proposition 2.4.1. *Let P be a module. Then P is a projective module if and only if P is isomorphic to a direct summand of a free module.*

Proposition 2.4.2. *Let P be a projective module. Then N is a projective module for any direct summand N of P.*

Proposition 2.4.3. *Let M_i be a module for all $i \in \{1, \dots, n\}$. Then M_i is a projective module for all $i \in \{1, \dots, n\}$ if and only if $M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is a projective module.*

Proposition 2.4.4. *Let P and M be modules and M be a projective modules. Then $\ker g$ is a direct summand of P for any epimorphism $g : P \to M$.*