โครงการ

การเรียนการสอนเพื่อเสริมประสบการณ์

ชื่อโครงการ จำนวนเชิงการนับของเซตของการเรียงสับเปลี่ยนบนเซตที่มีจุดไม่ตรึงจำนวนจำกัด

The cardinality of the permutations on a set with finite non-fixed points

ชื่อนิสิต นายจักรกฤษณ์ นันทศรี **เลขประจำตัว** 5933506023

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์

สาขาคณิตศาสตร์

ปีการศึกษา 2562

คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

จำนวนเชิงการนับของเซตของการเรียงสับเปลี่ยนบนเซตที่มีจุดไม่ตรึงจำนวนจำกัด

นายจักรกฤษณ์ นันทศรี

โครงงานนี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรวิทยาศาสตรบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2562

THE CARDINALITY OF THE PERMUTATIONS ON A SET WITH FINITE NON-FIXED POINTS

Mr. Jukkrid Nuntasri

A Project Submitted in Partial Fulfillment of the Requirements

for the Degree of Bachelor of Science Program in Mathematics

Department of Mathematics and Computer Sciences

Faculty of Sciences

Chulalongkorn University

Academic Year 2019

Project Title	THE CARDINALITY OF THE PERMUTATIONS ON A SET WITH FINITE NON-FIXED POINTS
By	Mr. Jukkrid Nuntasri
Field of Study	Mathematics
Project Advisor	Associate Professor Pimpen Vejjajiva, Ph.D.

Accepted by the Department of Mathematics and Computer SciencesFaculty of Sciences, Chulalongkorn University in Partial Fulfillment of the Requirements for the Bachelor's Degreein 2301499 Senior project

.... Head of the Department of Mathematics and Computer Sciences

(Professor Kritsana Neammanee, Ph.D.)

PROJECT COMMITTEE

.... Project Advisor

(Associate Professor Pimpen Vejjajiva, Ph.D.)

.... Committee

(Associate Professor Phichet Chaoha, Ph.D.)

.... Committee

(Athipat Thamrongthanyalak, Ph.D.)

จักรกฤษณ์ นันทศรี: จำนวนเชิงการนับของเซตของการเรียงสับเปลี่ยนบนเซตที่มีจุดไม่ตรึงจำนวนจำกัด. (THE CARDINALITY OF THE PERMUTATIONS ON A SET WITH FINITE NON-FIXED POINTS) อ.ที่ปรึกษาโครงงานหลัก : รศ. ดร. พิมพ์เพ็ญ เวชชาชีวะ, 27 หน้า.

ในทฤษฎีเซตแซร์เมโล-แฟรงเคล (ZF) ที่มีสัจพจน์การเลือก (AC) เซตของสับเซตทั้งหมดของ X ซึ่งแทนด้วย $\mathcal{P}(X)$ และเซตของการเรียงสับเปลี่ยนทั้งหมดบน X ซึ่งแทนด้วย $S(X)$ มีขนาดเท่ากันสำหรับเซตอนันต์ X ใด ๆ Dawson และ Howard ได้แสดงว่า เมื่อปราศจากสัจพจน์การเลือก จะไม่สามารถสรุปความสัมพันธ์ใด ๆ ระหว่างขนาดของเซตทั้งสองได้ ต่อมา Halbeisen และ Shelah ได้แสดงใน ZF ว่า $|\text{fin}(X)| < |\mathcal{P}(X)|$ สำหรับทุกเซตอนันต์ X โดยที่ $\text{fin}(X)$ เป็นเซตของสับเซตจำกัดทั้งหมดของ X เมื่อมีสัจพจน์การเลือก เราได้ว่า $|\text{fin}(X)| = |S_{\text{fin}}(X)|$ สำหรับทุกเซตอนันต์ X โดยที่ $S_{\text{fin}}(X)$ เป็นเซตของการเรียงสับเปลี่ยนบน X ทั้งหมดที่มีจุดไม่ตรึงจำนวนจำกัด อย่างไรก็ตาม ในทางตรงข้ามกับความสัมพันธ์ระหว่าง $|\text{fin}(X)|$ และ $|\mathcal{P}(X)|$ Tachtsis ได้แสดงว่า ข้อความ $|S_{\text{fin}}(X)| \neq |S(X)|$ ไม่มีทางพิสูจน์ได้ใน ZF สำหรับเซตอนันต์ X ใด ๆ ในโครงงานนี้ เราศึกษาความสัมพันธ์ระหว่าง $|\text{fin}(X)|$ และ $|S_{\text{fin}}(X)|$ สำหรับเซตอนันต์ X โดยปราศจากสัจพจน์การเลือกและได้ให้เงื่อนไขที่ทำให้เซตดังกล่าวสามารถเปรียบเทียบกันได้

ภาควิชา	คณิตศาสตร์และวิทยาการคอมพิวเตอร์	ลายมือชื่อนิสิต	จักรกฤษณ์ ....
สาขาวิชา	คณิตศาสตร์	ลายมือชื่อ อ.ที่ปรึกษาหลัก	....
ปีการศึกษา	2562		

5933506023: MAJOR MATHEMATICS

KEYWORDS: AXIOM OF CHOICE / CARDINAL / PERMUTATION

JUKKRID NUNTASRI : THE CARDINALITY OF THE PERMUTATIONS ON A SET WITH FINITE NON-FIXED POINTS. ADVISOR : ASSOC. PROF. PIMPEN VEJJAJIVA, Ph.D., 27 pp.

In the Zermelo-Frankel set theory (ZF) with the Axiom of Choice (AC), the set of subsets of X, $\mathcal{P}(X)$, and the set of permutations on X, $S(X)$, have the same cardinality for any infinite set X. Dawson and Howard showed that without AC, we cannot conclude any relationship between these cardinals. Halbeisen and Shelah showed, in ZF, that $|\text{fin}(X)| < |\mathcal{P}(X)|$ for any infinite set X, where $\text{fin}(X)$ is the set of finite subsets of X. With AC, $|\text{fin}(X)| = |S_{\text{fin}}(X)|$ for any infinite set X, where $S_{\text{fin}}(X)$ is the set of permutations on X with finite non-fixed points. However, in contrast with the relation between $|\text{fin}(X)|$ and $|\mathcal{P}(X)|$, Tachtsis showed that $|S_{\text{fin}}(X)| \neq |S(X)|$ is not provable in ZF for an arbitrary infinite set X. In this project, we study relationship between $|\text{fin}(X)|$ and $|S_{\text{fin}}(X)|$ for an infinite set X in the absence of AC and give some conditions that make them comparable.

Department:	Mathematics and Computer Sciences	Student's Signature	....
Field of Study:	Mathematics	Advisor's Signature	....
Academic Year:	2019		

Acknowledgements

I would like to thank my advisor, Associate Professor Pimpen Vejjajiva, for guiding my project, giving some background knowledges and suggestions. Also I would like to thank Associate Professor Phichet Chaoha and Athipat Thamrongthanyalak for being members of the project committee. Moreover, I would like to thank His Majesty King Maha Vajiralongkorn Bodindradebayavarangkun for giving me the scholarship in two previous semesters. Finally, I would like to thank my family very much for supporting me. This project is funded by His Majesty King Maha Vajiralongkorn Bodindradebayavarangkun and Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University.

CONTENTS

Page

Chapter I

INTRODUCTION

From high school mathematics, we know that the cardinality of the power set of a finite set A with $|A| = n$ is 2^n and the cardinality of the permutations on A, denoted by $S(A)$, is $n!$ which is greater than 2^n for all natural numbers $n \geq 4$. Surprisingly, for the case that A is an infinite set, in the Zermelo-Fraenkel set theory (ZF) with the Axiom of Choice (AC), these two cardinals are equal. However, each of "$|S(A)| < |\mathcal{P}(A)|$", "$|\mathcal{P}(A)| < |S(A)|$", and "$|S(A)|$ and $|\mathcal{P}(A)|$ are not comparable" for some infinite set A is consistent with ZF.

We consider the set of all finite subsets of a set A, denoted by $\text{fin}(A)$, and the set of all permutations on A with finite non-fixed points, denoted by $S_{\text{fin}}(A)$. In fact, $|A| = |\text{fin}(A)| = |S_{\text{fin}}(A)|$ for all infinite sets A in the Zermelo-Fraenkel set theory with Axiom of Choice (ZFC).

By the well-known Cantor's theorem, which is provable in ZF, we know that $|A| < |\mathcal{P}(A)|$ for any set A. Thus, in ZFC, for all infinite sets A,

$$|A| = |\text{fin}(A)| = |S_{\text{fin}}(A)| < |\mathcal{P}(A)| = |S(A)|.$$

Without AC, $|A| \leq |\text{fin}(A)|$ for all sets A but "$|A| = |\text{fin}(A)|$ for all infinite sets A" cannot be proved (see [4]). In 1994, Halbeisen and Shelah improved Cantor's theorem for infinite sets by showing, in ZF, that $|\text{fin}(A)| < |\mathcal{P}(A)|$ for all infinite sets A (see [3]). On the other hand, "$|S_{\text{fin}}(A)| \neq |S(A)|$ for all infinite sets A" is not provable in ZF (see [6]).

In this project, we give some conditions of an infinite set X as well as some weak forms of AC that make $|\text{fin}(X)|$ and $|S_{\text{fin}}(X)|$ comparable in ZF. First, we give some background about set theory in Chapter II. New results are in Chapter III.

Chapter II

PRELIMINARIES

At first, G. Cantor defined sets as collections of objects but this leads to paradoxes. One way to avoid these problems is to use an axiomatic method and let sets be undefined. This system is called an axiomatic set theory. Nowadays, Zermelo-Frankel set theory (ZF) with the axiom of choice (AC) is the most accepted axiomatic set theory. In this project, our work is done in ZF.

In this chapter, we give some background on set theory. Proofs of all theorems will be omitted. They can be found in [2].

2.1 Cardinal numbers

Intuitively, a *cardinal (number)* is a number used to measure the size of a set, i.e. the number of all elements of a set. Denote the *cardinal number* of a set X by $|X|$. Cardinals are defined so that for any sets X and Y, $|X| = |Y| \leftrightarrow X \approx Y$, where $X \approx Y$ means there is a bijection from X onto Y.

Definition. *Natural numbers* are constructed as follows:

$$0 = \emptyset, 1 = \{0\}, 2 = \{0, 1\}, 3 = \{0, 1, 2\}, \ldots$$

Let ω be the set of all natural numbers.

Definition. Let X be a set. If $X \approx n$ for some $n \in \omega$, X is said to be *finite* and define $|X| = n$. If X is not finite, then X is said to be *infinite*. We call $|X|$ a *finite cardinal* if X is finite; otherwise, $|X|$ is an *infinite cardinal*.

Note. Every finite cardinal is a natural number and vice-versa.

Notation. For any sets X and Y, let

1. $\mathcal{P}(X)$ be the set of all subsets of X,
2. $\text{fin}(X)$ be the set of all finite subsets of X,
3. $S(X) = \{f | f : X \to X \text{ is a bijection}\}$,
4. $S_{\text{fin}}(X) = \{f | f : X \to X \text{ is a bijection with finite non-fixed points}\}$,
5. $X^Y = \{f \mid f : Y \to X\}$.

Lemma 2.1.1. *For any cardinals κ and λ, there are sets X and Y such that $|X| = \kappa$, $|Y| = \lambda$, and $X \cap Y = \emptyset$.*

Definition. Let X and Y be sets and $\kappa = |X|$ and $\lambda = |Y|$. Define

1. $\kappa + \lambda = |X \cup Y|$ where $X \cap Y = \emptyset$,
2. $\kappa \cdot \lambda = |X \times Y|$,
3. $\kappa^\lambda = |X^Y|$.

Note. From the above definition, we have that for any cardinal κ and any natural number n,

$$\kappa^0 = 1 \text{ and } \kappa^{n+1} = \kappa \cdot \kappa^n.$$

Theorem 2.1.2. *For any set X, $|\mathcal{P}(X)| = 2^{|X|}$.*

Theorem 2.1.3. *Let κ, λ, and μ be cardinals. Then*

1. $\kappa + \lambda = \lambda + \kappa$,
2. $(\kappa + \lambda) + \mu = \kappa + (\lambda + \mu)$,
3. $\kappa \cdot \lambda = \lambda \cdot \kappa$,
4. $(\kappa \cdot \lambda) \cdot \mu = \kappa \cdot (\lambda \cdot \mu)$,
5. $\kappa \cdot (\lambda + \mu) = (\kappa \cdot \lambda) + (\kappa \cdot \mu)$,

6. $\kappa^{\lambda+\mu} = \kappa^{\lambda} \cdot \kappa^{\mu}$,

7. $(\kappa \cdot \lambda)^{\mu} = \kappa^{\mu} \cdot \lambda^{\mu}$,

8. $(\kappa^{\lambda})^{\mu} = \kappa^{\lambda \cdot \mu}$.

Definition. Let X and Y be sets and $\kappa = |X|$ and $\lambda = |Y|$. Then we say

1. $\kappa \leq \lambda$ if there is an injection from X into Y,

2. $\kappa < \lambda$ if $\kappa \leq \lambda$ but $\kappa \neq \lambda$.

Theorem 2.1.4. *$\leq$ partially orders the cardinal numbers.*

Theorem 2.1.5. *Let κ, λ, and μ be cardinals such that $\kappa \leq \lambda$*

1. $\kappa + \mu \leq \lambda + \mu$,

2. $\kappa \cdot \mu \leq \lambda \cdot \mu$,

3. $\kappa^{\mu} \leq \lambda^{\mu}$,

4. $\mu^{\kappa} \leq \mu^{\lambda}$, *if* $\kappa \neq 0$ *or* $\mu \neq 0$.

Theorem 2.1.6. *Cantor's theorem*
For any cardinal κ,

$$\kappa < 2^{\kappa}.$$

Theorem 2.1.7. *For all natural number $n \geq 4$,*

$$|\mathcal{P}(X)| = 2^n < n! = |S(X)|,$$

where X is a set with $|X| = n$.

2.2 Axiom of Choice

Definition. A *choice function* f for a set X is a function $f : X \backslash \{\emptyset\} \to \bigcup X$ such that for any $x \in X \backslash \{\emptyset\}$, $f(x) \in x$.

The following statements are equivalent forms of the Axiom of Choice (AC).

1. Well-ordering Theorem: Every set can be well-ordered.

2. Cardinal Comparability: For any cardinal numbers κ and λ, $\kappa \leq \lambda$ or $\lambda \leq \kappa$.

3. Every set has a choice function.

4. For every infinite cardinal κ, $\kappa^2 = \kappa$.

The following theorems are consequences of AC.

Theorem 2.2.1. *Absorption Law of Arithmetic*
For any cardinals κ and λ of which at least one is infinite,

1. $\kappa + \lambda = \max\{\kappa, \lambda\}$,

2. $\kappa \cdot \lambda = \max\{\kappa, \lambda\}$ *if* $\min\{\kappa, \lambda\} \neq 0$.

Theorem 2.2.2. *For any cardinals κ and λ, if X is a set such that $|X| = \kappa$ and each element of X has cardinality less than or equal to γ, then*

$$|\bigcup X| \leq \kappa \cdot \gamma.$$

Theorem 2.2.3. *Let X be an infinite set. Then*

$$|\mathcal{P}(X)| = |S(X)|.$$

Theorem 2.2.4. *Let X be an infinite set. Then*

$$|X| = |\text{fin}(X)| = |S_{\text{fin}}(X)|.$$

Corollary 2.2.5. *Let X be an infinite set. Then*

$$|X| = |\text{fin}(X)| = |S_{\text{fin}}(X)| < |\mathcal{P}(X)| = |S(X)|.$$

More details on AC can be found in [6].

2.3 Cardinal numbers without AC

Definition. An *aleph* is the cardinal of an infinite well-ordered set.

Note that, with AC, every set can be well-ordered. Therefore AC is equivalent to "every infinite cardinal is an aleph". In the absence of AC, any two alephs are comparable and satisfy absorption law of arithmetic.

Definition. Let $\aleph_0 = |\omega|$ and $\aleph_1$ be the least aleph greater than $\aleph_0$.

Definition. A set X is *countable* if $|X| \leq \aleph_0$. A set X is *denumerable* or *countably infinite* if $|X| = \aleph_0$

Theorem 2.3.1. *1.* $|\text{fin}(\omega)| = |S_{\text{fin}}(\omega)| = \aleph_0$.

2. $\aleph_0 + \aleph_0 = \aleph_0 \cdot \aleph_0 = \aleph_0$.

Cardinal Comparability is equivalent to AC. Therefore, without AC, we cannot guarantee whether two cardinals are comparable or not, in particular, infinite cardinals may not be compared with $\aleph_0$.

Definition. A set X is called *Dedekind-infinite* if $\aleph_0 \leq |X|$; otherwise, X is called a *Dedekind-finite* set.

Note. Every Dedekind-infinite set is infinite but the converse is not necessarily true without AC.

2.4 Weak forms of AC

Even though AC is equivalent to many important theorems, for example, Zorn's lemma, Tychonoff's theorem, and "every vector space has a basis", it also leads to some counterintuitive results such as Banach-Tarski paradox. Thus some mathematicians avoid using AC and sometimes use weaker forms of AC instead.

The following weak choice principles are relevant to our work.

1. $\text{AC}_{<\aleph_0}$: Every family of finite sets has a choice function.

2. D-fin: Every infinite set is Dedekind-infinite.

3. $2\mathfrak{m} = \mathfrak{m}$: For any infinite cardinal $\mathfrak{m}$, $2\mathfrak{m} = \mathfrak{m}$.

Relations among these weak forms are as follows:

- $2\mathfrak{m} = \mathfrak{m}$ implies D-fin but the converse is not provable in ZF. In other words, D-fin is weaker than $2\mathfrak{m} = \mathfrak{m}$.

- It is not provable in ZF that $AC_{<\aleph_0}$ implies D-fin. As a result, "$AC_{<\aleph_0}$ implies $2\mathfrak{m} = \mathfrak{m}$" is not provable as well.

- It is unknown whether $2\mathfrak{m} = \mathfrak{m}$ implies $AC_{<\aleph_0}$ or not.

The above results are from [5].

Chapter III

MAIN RESULTS

In [1], Dawson and Howard proved that for every set X, if $|X| = 2|X|$, then $|\mathcal{P}(X)| \leq |S(X)|$ and if $|X| = |X|^2$, then $|\mathcal{P}(X)| \geq |S(X)|$. Since "$|X| = |X|^2$ for any infinite set X" is an equivalent form of AC and "$|X| = 2|X|$ for any infinite set X" is a weaker form of AC, if AC is assumed, $|\mathcal{P}(X)| = |S(X)|$ for any infinite set X.

However, they also proved that each of "$|S(X)| < |\mathcal{P}(X)|$", "$|\mathcal{P}(X)| < |S(X)|$", and "$|S(X)|$ and $|\mathcal{P}(X)|$ are not comparable" for some infinite set X is consistent with ZF. Thus, without AC, we cannot conclude any relationship between $\mathcal{P}(X)$ and $S(X)$ for an arbitrary infinite set X.

In [3], Halbeisen and Shelah improved Cantor's theorem for infinite sets by showing, in ZF, that $|\text{fin}(X)| < |\mathcal{P}(X)|$ for all infinite sets X. This implies that, in the absence of AC, $|X| \leq |\text{fin}(X)| < |\mathcal{P}(X)|$ for all infinite sets X. In [4], they also showed that the statement "$|X| = |\text{fin}(X)|$ for all infinite sets X" cannot be proved in ZF.

On the other hand, in [7], Tachtsis proved that the statement "$|S_{\text{fin}}(A)| \neq |S(A)|$ for all infinite sets A" is not provable in ZF.

In this chapter, we shall give conditions that make two cardinals $|\text{fin}(X)|$ and $|S_{\text{fin}}(X)|$ comparable for an infinite set X.

First, we give conditions that make $|\text{fin}(X)| \leq |S_{\text{fin}}(X)|$ for an infinite set X. For the first condition, we need the following definitions.

Definition. For a function f on a set X, we define $f^n : X \to X$ recursively by

$$f^0 = \text{id}_X \text{ and } f^{n+1} = f \circ f^n.$$

Definition. For a bijection f on a set X, we define $f^{-n} = (f^{-1})^n$ for all $n \in \omega$.

Definition. For a bijection f on a set X, we define a relation $\sim_f$ on X as follows:

$$a \sim_f b \text{ iff } b = f^n(a) \text{ for some } n \in \mathbb{Z}.$$

Note. For any bijection f on a set X, $\sim_f$ is an equivalence relation on X. Moreover, for $x \in X$, if $[x]_{\sim_f}$ is finite, then $[x]_{\sim_f} = \{x, f(x), ..., f^n(x)\}$ for some $n \in \omega$ and if $[x]_{\sim_f}$ is infinite, then $[x]_{\sim_f} = \{f^n(x)|n \in \mathbb{Z}\}$, so $|[x]_{\sim_f}| = \aleph_0$.

Notation. Let X be a set, $n \in \omega$ be a natural number greater than 1, and $x_1, x_2, ..., x_n$ be distinct elements of X. We write $(x_1; x_2; ...; x_n)$ for the cyclic permutation

$$\{(x_1, x_2), (x_2, x_3), ..., (x_n, x_1)\} \cup \mathrm{id}_{X \setminus \{x_1,...,x_n\}}.$$

Lemma 3.1. *Let X be an infinite set. If there is a bijection $f : X \to X$ such that f^2 and f^3 have finite fixed points, then there is a bijection $g : X \to X$ such that g^2 and g^3 have no fixed points.*

Proof. Assume there exists a bijection f as in the theorem. Let $F = \{a \in X|f^2(a) = a$ or $f^3(a) = a\}$. Since F is finite and X is infinite, there is a class $[x]_{\sim_f}$ such that $|[x]_{\sim_f}| \geq 4$. Then, $[x]_{\sim_f} \cap F = \emptyset$. Since $[x]_{\sim_f}$ is countable, $4 \leq |[x]_{\sim_f} \cup F| \leq \aleph_0$.

Case 1: $|[x]_{\sim_f} \cup F| < \aleph_0$.

Let $\{a_1, a_2, ..., a_n\}$ be an enumeration of $[x]_{\sim_f} \cup F$. Define

$$g = (f \upharpoonright (X \setminus \{a_1, ..., a_n\}) \cup \mathrm{id}_{[x]_{\sim_f} \cup F}) \circ (a_1; a_2; ...; a_n).$$

Clearly, g^2 and g^3 have no fixed points.

Case 2 $|[x]_{\sim_f} \cup F| = \aleph_0$.

Let $\{a_i|i \in \mathbb{Z}\}$ be an enumeration of $[x]_{\sim_f} \cup F$. Define $g : X \to X$ by $g(x) = f(x)$ if $x \notin \{a_i|i \in \mathbb{Z}\}$ and $g(a_i) = a_{i+1}$ for all $i \in \mathbb{Z}$. Clearly, g^2 and g^3 have no fixed points. □

Theorem 3.2. *Let X be an infinite set. If there exists a bijection $f : X \to X$ such that f^2 and f^3 have finite fixed points, then $|\mathrm{fin}(X)| \leq |S_{\mathrm{fin}}(X)|$.*

Proof. Assume there exists a bijection f as in the theorem. By Lemma 3.1, there exists a bijection $g : X \to X$ such that g^2 and g^3 have no fixed points. We write $[x]_{\sim_g}$ as $[x]$ for all $x \in X$. Note that, since g^2 and g^3 have no fixed points, $|[x]| \geq 4$ for all $x \in X$.

For each $x \in X$, we define a function $g_{[x]} : \mathrm{fin}([x]) \to S_{\mathrm{fin}}([x])$ by

$$g_{[x]}(\emptyset) = \mathrm{id}_{[x]}$$

$$g_{[x]}(\{a\}) = (g(a); g^3(a); g^2(a))$$
$$g_{[x]}(\{a, g^{n_1}(a), ..., g^{n_l}(a)\}) = (a; g^{n_1}(a); ...; g^{n_l}(a)),$$

where $1 \leq l < |[x]|$ and $1 \leq n_1 < n_2 < ... < n_l < |[x]|$.

Claim 1: $g_{[x]}$ is well-defined for each $x \in X$.

Let $x \in X$ and $Y \in \text{fin}([x])$. If Y is $\emptyset$ or a singleton, it is clear that $g_{[x]}(Y)$ can be uniquely determined.

For the last case, $|Y| = l + 1$ for some $1 \leq l < |[x]|$ and let $Y = \{a, a_1, ..., a_l\} = \{b, b_1, ..., b_l\}$, where $a, a_1, ..., a_l$ and $b, b_1, ..., b_l$ are two distinct enumerations of Y. Since for any natural number $1 \leq i \leq l$, a and a_i are in the same equivalence class, $a_i = g^{n'_i}(a)$ for some $n'_i < |[x]|$. Now, we arrange $n'_1, ..., n'_l$ in an increasing order, say $n_1, ..., n_l$. Therefore, $Y = \{a, g^{n_1}(a), ..., g^{n_l}(a)\}$ where $1 \leq n_1 < n_2 < ... < n_l < |[x]|$. Similarly for the set $\{b, b_1, ..., b_l\}$, $Y = \{b, g^{m_1}(b), ..., g^{m_l}(b)\}$ for some $1 \leq m_1 < m_2 < ... < m_l < |[x]|$.

Since $b \in Y$ and $a, a_1, ..., a_l$ and $b, b_1, ..., b_l$ are distinct enumerations, there exists a natural number $0 < k \leq l$ such that $b = g^{n_k}(a)$. Note that for any $i \leq l$, among the elements of Y, $g^{n_{i+1}}(a)$ appears first in the sequence $g^{n_i+1}(a), g^{n_i+2}(a), ...$, similarly for $g^{m_{j+1}}(b)$ (here, $n_0 = 0 = m_0$). Since $g^{n_k}(a) = b$, the sequences $g^{n_k+1}(a), g^{n_k+2}(a), ...$ and $g^{m_0+1}(b), g^{m_0+2}(b), ...$ are identical, so $g^{n_{k+1}}(a) = g^{m_1}(b)$. By induction, we can show that $g^{n_{k+i}}(a) = g^{m_i}(b)$ for all $i \leq l - k$. Since $k \neq 0$, $a \in Y \backslash \{b, g^{m_1}(b), ..., g^{m_{l-k}}(b)\}$, so $a = g^{m_r}(b)$ for some $l - k + 1 \leq r \leq l$. By the same process as described above, we have that $g^{n_i}(a) = g^{m_{r+i}}(b)$ for all $i < k$ where $r + i \leq l$. Hence $r \leq l - k + 1$, so $r = l - k + 1$. Thus the sequences $a, g^{n_1}(a), ..., g^{n_l}(a)$ and $g^{m_{l-k+1}}(b), g^{m_{l-k+2}}(b), ..., g^{m_l}(b), b, g^{m_1}(b), ..., g^{m_{l-k}}(b)$ are identical. Hence,

$$\begin{aligned}(a; g^{n_1}(a); ...; g^{n_l}(a)) &= (g^{m_{l-k+1}}(b); g^{m_{l-k+2}}(b); ...; g^{m_l}(b); b; g^{m_1}(b); ...; g^{m_{l-k}}(b)) \\ &= (b; g^{m_1}(b); ...; g^{m_l}(b)).\end{aligned}$$

□

Claim 2: $g_{[x]}$ is injective for each $x \in X$.

Let $Y, Z \in \text{fin}([x])$ be such that $Y \neq Z$. Without loss of generality, assume there is a $y \in Y \backslash Z$. If $Z = \emptyset$, then $g_{[x]}(Z) = \text{id}_{[x]} \neq g_{[x]}(Y)$.

Now, suppose $Z \neq \emptyset$.

<u>Case 1</u> $Y = \{y\}$.

Then $g_{[x]}(Y) = (g(y); g^3(y); g^2(y))$.

<u>Case 1.1</u> $Z = \{z\}$.

Then $y \neq z$ and $g_{[x]}(Z) = (g(z); g^3(z); g^2(z))$. Since g is injective, $g(y) \neq g(z)$. If $y = g(z)$, then $g_{[x]}(Z)(y) = g^3(z) = g^2(y)$. If $y = g^2(z)$, then $g_{[x]}(Z)(y) = g(z)$. Since g^2 (and hence g) has no fixed points, from both cases, $g_{[x]}(Y)(y) = y \neq g_{[x]}(Z)(y)$. Otherwise $g(y) \notin \{g(z), g^2(z), g^3(z)\}$. Hence $g_{[x]}(Y)(g(y)) = g^3(y) \neq g(y) = g_{[x]}(Z)(g(y))$.

<u>Case 1.2</u> $|Z| > 1$.

Then $g_{[x]}(Z)$ is a cyclic permutation which permutes every element in Z.

If $Z = \{g(y), g^2(y), g^3(y)\}$, then $g_{[x]}(Y)(g(y)) = g^3(y) \neq g^2(y) = g_{[x]}(Z)(g(y))$.

Otherwise, clearly $g_{[x]}(Y) \neq g_{[x]}(Z)$.

<u>Case 2</u> $|Y| > 1$.

As in the case 1.2, if $|Z| = 1$, then $g_{[x]}(Y) \neq g_{[x]}(Z)$.

Suppose $|Z| > 1$. Since $y \in Y \setminus Z$ where $g_{[x]}(Y)$ permutes every element in Y, $g_{[x]}(Y)(y) \neq y = g_{[x]}(Z)(y)$. Hence $g_{[x]}(Y) \neq g_{[x]}(Z)$. □

We define a function $G : \text{fin}(X) \to S_{\text{fin}}(X)$ by $G(\emptyset) = \text{id}_X$ and

$$G(S) = g_{[s_1]}(S \cap [s_1]) \circ g_{[s_2]}(S \cap [s_2]) \circ ... g_{[s_n]}(S \cap [s_n]),$$

where $S \neq \emptyset$ and $\{[x] \in X/_\sim | [x] \cap S \neq \emptyset\} = \{[s_1], [s_2], ..., [s_n]\}$. Since for $1 \leq i \leq k$, $g_{[s_i]}$ only permutes finitely many elements in the class $[s_i]$ and these classes are pairwise disjoint, G is well-defined. Finally, we show that G is an injection.

Let $A, B \in \text{fin}(X)$ be such that $A \neq B$. Without loss of generality, assume there is $a \in A \setminus B$. If $B = \emptyset$, then $G(B) = \text{id}_X \neq G(A)$.

Now, suppose $B \neq \emptyset$.

<u>Case 1</u> $[a] \cap B = \emptyset$.

Then $G(A) \upharpoonright [a] = g_{[a]}(A \cap [a]) \neq id_{[a]} = G(B) \upharpoonright [a]$. So, $G(A) \neq G(B)$.

<u>Case 2</u> $[a] \cap B \neq \emptyset$.

Let $b \in [a] \cap B$. Note that $a \in (A \cap [a]) \setminus (B \cap [a])$. Since $g_{[a]}$ is injective, $G(A) \upharpoonright [a] = g_{[a]}(A \cap [a]) \neq g_{[a]}(B \cap [a]) = G(B) \upharpoonright [a]$. So, $G(A) \neq G(B)$. □

Corollary 3.3. *Under* $2\mathfrak{m} = \mathfrak{m}$, $|\text{fin}(X)| \leq |S_{\text{fin}}(X)|$ *for any infinite set* X.

Proof. Assume $2\mathfrak{m} = \mathfrak{m}$ and let X be an infinite set. Then $|X|$ is an infinite cardinal and $|4 \times X| = 4|X| = 2(2|X|) = 2|X| = |X|$. Let $g : 4 \times X \to X$ be a bijection. Define $f : X \to X$ by $f(x) = (g\circ((0, y); ...; (3, y))\circ g^{-1})(x)$ if $x = g(k, y)$ for some $(k, y) \in 4\times X$. That is $f(g(k, y)) = g(k + 1, y)$ for all $k < 3$, and $f(g(3, y)) = g(0, y)$ for all $y \in X$. Then f is a bijection and for each $x \in X$, $[x]_{\sim_f} = \{g(0, y), g(1, y), g(2, y), g(3, y)\}$ if $x = g(k, y)$ for some $k < 4$ and $y \in X$. Since $|[x]_{\sim_f}| = 4$ for all $x \in X$, f^2 and f^3 have no fixed points. By Theorem 3.2, $|\text{fin}(X)| \leq |S_{\text{fin}}(X)|$ as desired. $\square$

Next, we shall give another result.

Theorem 3.4. *Let X be an infinite set. If* $\text{fin}(X)$ *has a choice function, then* $|\text{fin}(X)| \leq |S_{\text{fin}}(X)|$.

Proof. Let X be an infinite set such that $\text{fin}(X)$ has a choice function. Let $Y = \{y_0, y_1, y_2, y_3\}$ be a subset of X where y_0, y_1, y_2, y_3 are distinct. Since $\text{fin}(X)$ has a choice function, say F, every $A \in \text{fin}(X)$ has a linear order $<_A$ induced by the ordering on ω via the map $\phi_A : |A| \to A$ defined recursively by $\phi_A(k) = F(A\backslash\phi_A[k])$.

Define $\Pi : \text{fin}(X) \to S_{\text{fin}}(X)$ by

$$\Pi(A) = \begin{cases} \text{id}_X & \text{if } A = \emptyset; \\ (a_0; a_1; ...; a_{|A|-1}) & \text{if } |A| > 1 \text{ and } A = \{a_0, a_1, ..., a_{|A|-1}\} \\ & \text{where } a_i <_A a_{i+1} \text{ for all } i < |A| - 1; \\ (\Pi(Y\backslash\{y_i\}))^{-1} & \text{if } A = \{y_i\} \text{ for some } i \in \{0, 1, 2, 3\}; \\ (\Pi(\{y_1, y_2, a\}))^{-1} & \text{if } A = \{a\} \text{ for some } a \in X\backslash Y. \end{cases}$$

Note that since $\Pi(Y\backslash\{y_i\})$, where $i < 4$, is a cycle of length 3, $\Pi(Y\backslash\{y_i\}) \neq (\Pi(Y\backslash\{y_i\}))^{-1}$. Similarly for $\Pi(\{y_1, y_2, a\})$ where $a \in X\backslash Y$. It is left to show that Π is an injection. Let $A, B \in \text{fin}(X)$ be such that $\Pi(A) = \Pi(B) = \pi$. If $\pi = \text{id}_X$, then $A = B = \emptyset$. Suppose $\pi \neq \text{id}_X$. Then $M = \{x \in X : \pi(x) \neq x\} \neq \emptyset$. We distinguish into cases.

<u>Case 1</u> $M = Y\backslash\{y_i\}$ for some natural number $i < 4$. Then $\pi = (\Pi(Y\backslash\{y_i\}))^{-1} = \Pi(\{y_i\})$ or $\pi = \Pi(Y\backslash\{y_i\})$. Since $(\Pi(Y\backslash\{y_i\}))^{-1} \neq \Pi(Y\backslash\{y_i\})$, $A = \{y_i\} = B$ or $A = Y\backslash\{y_i\} = B$.

Case 2 $M = \{y_1, y_2, a\}$ for some $a \in X \backslash Y$. Then $\pi = (\Pi(\{y_1, y_2, a\}))^{-1} = \Pi(\{a\})$ or $\pi = \Pi(\{y_1, y_2, a\})$. Similar to the above case, $A = \{a\} = B$ or $A = \{y_1, y_2, a\} = B$.

Otherwise, we have $\pi = \Pi(M)$ so $A = B = M$ as desired. □

For the other direction, we need a stronger condition than that in the previous theorem.

Theorem 3.5. *Let X be a Dedekind-infinite set. If* $\text{fin}(X)$ *has a choice function, then* $|S_{\text{fin}}(X)| \leq |\text{fin}(X)|$.

Proof. Let X be a Dedekind-infinite set such that $\text{fin}(X)$ has a choice function. Since $\omega \times S_{\text{fin}}(\omega) \approx \omega \times \omega \approx \omega$, there exists an injection $g : \omega \times S_{\text{fin}}(\omega) \to X$. Let $G = g[\omega \times S_{\text{fin}}(\omega)]$ and $a_{n,\sigma} = g(n, \sigma)$ for all $n \in \omega$ and $\sigma \in S_{\text{fin}}(\omega)$. For each $\pi \in S_{\text{fin}}(X)$, define $m(\pi) = \{x \in X : \pi(x) \neq x\}$.

For each $A \in \text{fin}(X)$, let $\phi_A : |A| \to A$ be defined as in the proof of Theorem 3.4. Recall that ϕ_A is a bijection which induces a linear order $<_A$ on A for each $A \in \text{fin}(X)$. Finally, for $\pi \in S_{\text{fin}}(X) \backslash \{\text{id}_X\}$, define $\pi^\circ = (\phi^{-1}_{m(\pi)} \circ \pi \circ \phi_{m(\pi)}) \cup \text{id}_{\omega \backslash |m(\pi)|}$. Then $\pi^\circ : \omega \to \omega$ is a bijection. Since for any $\pi \in S_{\text{fin}}(X)$, $m(\pi) \in \text{fin}(X)$ and hence $\pi^\circ \in S_{\text{fin}}(\omega)$. Note that for any distinct $\pi, \psi \in S_{\text{fin}}(X) \backslash \{id_X\}$, it is possible that $\pi^\circ = \psi^\circ$ but if $m(\pi) \neq m(\psi)$, then $\pi^\circ \neq \psi^\circ$.

Define $f : S_{\text{fin}}(X) \to \text{fin}(X)$ by $f(\text{id}_X) = \emptyset$ and $f(\pi) = m(\pi) \cup \{a_{M,\pi^\circ}\}$ for all $\pi \in S_{\text{fin}}(X) \backslash \{\text{id}_X\}$ where

$$M = \begin{cases} 0 & \text{if } m(\pi) \cap G = \emptyset; \\ \max\{n \in \omega : \exists \sigma \in S_{\text{fin}}(\omega)(a_{n,\sigma} \in m(\pi))\} + 1 & \text{otherwise.} \end{cases}$$

We will show that f is injective. Let $\pi, \psi \in S_{\text{fin}}(X)$ be such that $f(\pi) = f(\psi) = F$. If $F = \emptyset$, then $\pi = \psi = \text{id}_X$. If $F \neq \emptyset$, then $F \cap G \neq \emptyset$. Let K be such largest $k \in \omega$ such that there exists $\sigma \in S_{\text{fin}}(\omega)$ where $a_{k,\sigma} \in F$. Then $m(\pi) \cup \{a_{K,\pi^\circ}\} = m(\psi) \cup \{a_{K,\psi^\circ}\}$. Note that for any $l \in \omega$ and $\rho \in S_{\text{fin}}(\omega)$, if $a_{l,\rho} \in m(\pi) \cup m(\psi)$, then $l < k$. Hence $a_{k,\pi^\circ} = a_{k,\psi^\circ}$, so $\pi^\circ = \psi^\circ = \sigma$ and $m(\pi) = F \backslash \{a_{K,\sigma}\} = m(\psi)$.

This implies that $\pi(x) = x = \psi(x)$ for all $x \in X \backslash m(\pi)$. It is left to show that $\pi(x) = \psi(x)$ for all $x \in m(\pi)$. Let $x \in m(\pi)$. Then $\phi^{-1}_{m(\pi)}(x) < |m(\pi)|$ and

$$\pi(x) = \phi_{m(\pi)} \circ \pi^\circ \circ \phi^{-1}_{m(\pi)}(x) = \phi_{m(\psi)} \circ \psi^\circ \circ \phi^{-1}_{m(\psi)}(x) = \psi(x).$$

We have $\pi = \psi$ as desired. □

Thus, we can conclude the above results in the following corollary.

Corollary 3.6. *If we assume* $\mathrm{AC}_{<\aleph_0}$*, then* $|\mathrm{fin}(X)| = |S_{\mathrm{fin}}(X)|$ *for any Dedekind infinite set* X*. That is, under* $\mathrm{AC}_{<\aleph_0}$ *and D-fin,* $|\mathrm{fin}(X)| = |S_{\mathrm{fin}}(X)|$ *for any infinite set* X.

REFERENCES

[1] J. W. Dawson, Jr. and P. E. Howard. Factorial of infinite cardinals, pages 185–195. Fundamenta Mathematicae 93, 1976.

[2] H. B. Enderton. Elements of set theory. Academic Press, 1977.

[3] L. J. Halbeisen and S. Shelah. Consequences of Arithmetic for set theory, pages 30–40. Journal of Symbolic Logic 59, 1994.

[4] L. J. Halbeisen and S. Shelah. Relations between Some Cardinals in the Absence of the Axiom of Choice, pages 237–261. Bulletin of Symbolic Logic 7, 2001.

[5] P. Howard and J. E. Rubin. Consequences of the Axiom of Choice, volume 59. Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 1998.

[6] T. J. Jech. The Axiom of Choice. New York: Dover Publications, 2008.

[7] E. Tachtsis. On the existence of permutations of infinite set without fixed points in set theory without choice, pages 281–300. Acta Mathematica Hungarica 157, 2019.

Appendix I

The Project Proposal of Course 2301399 Project Proposal Academic Year 2019

Project Title (Thai)	จำนวนเชิงการนับของเซตของการเรียงสับเปลี่ยนบนเซต ที่มีจุดไม่ตรึงจำนวนจำกัด
Project Title (English)	The cardinality of the permutations on a set with finite non-fixed points.
Project Advisor	Assoc. Prof. Dr. Pimpen Vejjajiva
By	Mr. Jukkrid Nuntasri ID 5933506023 Mathematics, Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University.

Background and Rationale

In high school mathematics, everyone knows that the cardinality of the power set of a finite set A with $|A| = n$ is 2^n and the cardinality of the permutations on A, denoted by $S(A)$, is $n!$ which is greater than 2^n for all natural numbers $n \geq 4$. Surprisingly, for the case that A is an infinite set, in the Zermelo-Fraenkel set theory (ZF) with the Axiom of Choice (AC), these two cardinals are equal. However, each of "$|S(A)| < |\mathcal{P}(A)|$", "$|\mathcal{P}(A)| < |S(A)|$", and "$|S(A)|$ and $|\mathcal{P}(A)|$ are not comparable" for some infinite set A is consistent with ZF.

We consider the set of all finite subsets of a set A, denoted by $\text{fin}(A)$ and the set of all permutations on A with finite non-fixed points, denoted by $S_{\text{fin}}(A)$. In fact, $|A| = |\text{fin}(A)| = |S_{\text{fin}}(A)|$ for all infinite sets A in the Zermelo-Fraenkel set theory with Axiom of Choice (ZFC).

By the well-known Cantor's theorem, which is provable in ZF, we know that $|A| < |\mathcal{P}(A)|$ for any set A. Thus, in ZFC, for all infinite sets A,

$$|A| = |\text{fin}(A)| = |S_{\text{fin}}(A)| < |\mathcal{P}(A)| = |S(A)|.$$

Without AC, $|A| \leq |\text{fin}(A)|$ for all sets A but "$|A| = |\text{fin}(A)|$ for all infinite sets A" cannot be proved (see [4]). In 1994, Halbeisen and Shelah improved Cantor's theorem for infinite sets by showing, in ZF, that $|\text{fin}(A)| < |\mathcal{P}(A)|$ for all infinite sets A (see [3]). On the other hand, "$|S_{\text{fin}}(A)| \neq |S(A)|$ for all infinite sets A" is not provable in ZF (see [6]).

While relationships between $\mathcal{P}(A)$ and $S(A)$ for an infinite set A have been widely studied, those of $\text{fin}(A)$ and $S_{\text{fin}}(A)$ are still open. Thus, it is interesting to know, in the absence of AC, whether any relationship between them is provable or not.

Objective

Study relationships between $|\text{fin}(X)|$ and $|S_{\text{fin}}(X)|$ in ZF.

Scope

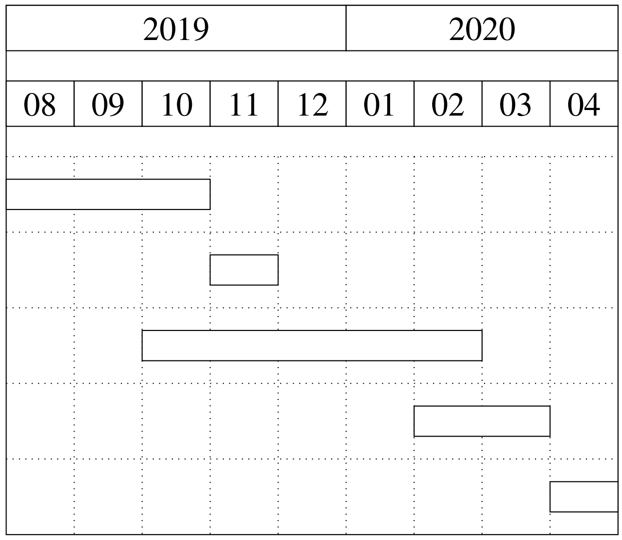
In this project, we work in ZF. For consistency results, we use permutation models.

Project Activities

1. Study AC and its consequences.
2. Study properties of cardinal numbers without AC.
3. Find conditions of an infinite set X that make $|\text{fin}(X)|$ and $|S_{\text{fin}}(X)|$ comparable in ZF.
4. Study permutation models.
5. Investigate some consistency results concerning $|\text{fin}(X)|$ and $|S_{\text{fin}}(X)|$.

Scheduled operations

1. Related research study
2. Project Proposal
3. Project conduct
4. Report and proofreading
5. Project presentation



Benefits

Obtain relations between $|\mathrm{fin}(X)|$ and $|S_{\mathrm{fin}}(X)|$ in the absence of AC.

Budget

1. Textbook 5000 Baht

References

[1] Dawson, J. W., Jr., Howard, Paul E., **Factorial of infinite cardinals**, Fundamenta Mathematicae, 1976, pp. 185-199.

[2] Enderton, H. B., **Elements of set theory**, Academic Press, 1977.

[3] Halbeisen, L., Shelah, S., **Consequences of Arithmetic for set theory**, Journal of Symbolic Logic, 1994, pp. 30-40.

[4] Halbeisen, L., Shelah, S., **Relations between Some Cardinals in the Absence of the Axiom of Choice**, Bulletin of Symbolic Logic 7, 2001, pp. 237-261.

[5] Jech, T. J., **The Axiom of Choice**. New York: Dover Publications, Inc., 2008.

[6] Tachtsis E., **On the existence of permutations of infinite set without fixed points in set theory without choice**, Acta Mathematica Hungarica, 2018, pp. 281-300.

Biography

นายจักรกฤษณ์ นันทศรี   รหัสนิสิต 5933506023
ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
สาขาคณิตศาสตร์ คณะวิทยาศาสตร์
จุฬาลงกรณ์มหาวิทยาลัย
นิสิตทุนสมเด็จพระบรมโอรสาธิราช เจ้าฟ้ามหาวชิราลงกรณ
สยามมกุฎราชกุมาร