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Twistor Theory and Scattering Amplitudes in super Yang-Mills Theory

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Twistor Theory and Scattering Amplitudes in super Yang-Mills Theory

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A report submitted to the Department of Physics of Chulalongkorn University
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Project Title:	Twistor Theory and Scattering Amplitudes in super Yang-Mills Theory
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Department:	Physics
Academic Year:	2020

This report is submitted to the Department of Physics, Faculty of Science, Chulalongkorn University, impartial fulfillment of the requirements for the degree of Bachelor of Science.

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ABSTRACT

In quantum field theory, the probability of outcomes of a scattering process can be calculated through scattering amplitudes. The scattering amplitudes can be calculated using the Feynman diagrams approach. However, the Feynman diagrams approach can become too difficult for beyond five particles. Focusing on the amplitude of Yang-Mills theory, the spinor-helicity formalism was used to help calculate the amplitude, resulting in a simple formula for an n particle scattering event with two negative helicity particles.

For general helicity configuration, we focus on the work of Witten and Roiban, Spradlin, and Volovich (RSVW) that incorporated twistor theory to their formulation. This results in a formula on twistor space for general helicity configuration of tree level gluon scattering, localized on constraint equations known as the refined scattering equations. The number of solutions to these constraints are observed to be the Eulerian numbers $E(n-3, k-2)$, where k is the number of negative helicity particles. In this project, we will review the formulation of the tree level amplitude including the spinor-helicity formalism, twistor theory, and RSVW formula. Then, we proved that the number of solutions to the constraints to RSVW formula are the Eulerian numbers by establishing a recursion relation for the number of solutions, using the method of dominance balance.

Recently, a twistor formula for one loop amplitude was published, and the number of solutions to these constraints – the loop polarized scattering equations– are still unknown. In order to be prepared for finding the number of solutions for the loop polarized scattering equations, we reproduce the proof for the number of solutions to the constraints of another one-loop formula, which is an extension of a formula proposed by Cachazo, He, and Yuan (CHY). The methods that were used to establish a recursion relation for the number of solutions to the one loop CHY's constraints were then applied to the loop polarized scattering equations. Up to this point, we found that this recursion relation possesses some similar features to those of one loop CHY constraints.

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Introduction

Physics is a study of natural phenomena by inferring a general theory from some observations. Not only that the theory must provide some explanation for the observed phenomena, it should be able to make some predictions for other phenomena too. Classical physics require measurements of position, momentum, and other measurable quantities to predict other unknown or un-measurable quantities. An outcome of a classical collision can be predict by knowing the mass and velocity of the incoming objects. However, quantum mechanical measurement is different.

Quantum mechanics describe nature as probabilistic and one can only measure the "chance" of several possible outcomes in an event. Knowing the position and momentum of an incoming quantum particle is not enough to exactly predict the outcome of a scattering event, we can only calculate the chances of outcome being so and so. The experimentally measurable quantities of quantum mechanical scattering events is the total cross-section σ, being the ratio of the number of particles scattered out (N) and the incoming flux Φ over a period of time T, i.e. $\sigma = N/(T\Phi)$, giving the outcome of the event at all angle [1]. Another quantity that can also be measured is the differential cross-section $d\sigma/d\Omega$, in which the measurement is done only in a certain solid angle Ω.

In quantum field theory, the differential cross-section can be calculated by looking at the quantity called the S-matrix, which is a matrix connecting the incoming and outgoing state in a given interaction. The S-matrix encoded the details of the interactions, whether it is possible and how likely it is to happen. The S-matrix is calculated perturbatively around a free theory, where particles have no interactions. The quantities that are calculated are then the non-trivial elements of the S-matrix, i.e. the event with interactions, called the scattering amplitudes.

The calculation of scattering amplitudes can be done by looking at the particular income and outcome and the interaction from the Lagrangian, or the correlation functions, which are integrals over Green functions determined from the Lagrangian. Feynman realized that these calculations can be pictorially represented by graphs, called the Feynman diagrams, where each external line, vertex, and internal line represent mathematical calculations.

Feynman diagrams are powerful tools in quantum field theory in calculating scattering amplitudes. But using them beyond 4-5 particles can cause the calculation to be a nightmare, as the number of possible diagrams increases drastically, to the point that it is impossible for humans and quite difficult for modern computers. In a scattering of gluons, which are gauge bosons described by Yang-Mills theory, a similar situation occurs that the number of diagrams grows drastically if there are more external particles [2]. There are so many diagrams required to be calculated, but what's special is that most diagrams vanish, and the final cross-section looks remarkably simple.

Complicated calculation simplifying into some elegant and compact equation leads to the doubt that maybe calculating the Yang-Mills amplitude using Feynman diagrams is hiding something and that there exists another simple way to look at these amplitudes. In [3], a simple formula was given by Parke and Taylor for an n particle scattering with two negative/positive and $n-2$ positive/negative helicity gluons. This formula uses the polarizations of the gluons and the decomposition of the spin-1 massless vector into the spinorial fundamental representation. This is called the spinor-helicity formalism [2],[4] .

General helicity configuration amplitudes can be studied by finding the connection of the general helicity as the Parke-Taylor formula, and there are methods that use supersymmetric calculation to study the NMHV. This project is interested in a method achieved in the work of Witten [5] and Roiban, Spradlin, and Volovich [6], that a general scattering amplitude of tree level Yang-Mills theory was given as a formula on a special space called twistor space, now known as the RSVW formula. Twistor theory was proposed by Roger Penrose in 1967 [7]. The theory provides a framework for describing information on 4d-spacetime as geometric objects on another space with some special properties called twistor space, such that there exists a one-to-one correspondence between the two spaces. The property that makes twistor space special is that it describes the conformal structure of a light cone with a holomorphic structure. With this correspondence, a point in twistor space corresponds with a light ray on spacetime, and a line in twistor space corresponds to a point in spacetime. This allows us to describe massless objects on spacetime in a simpler way on twistor space.

The RSVW formula is a formula on twistor space that is localized on some constraint equations. These constraint equations are known as the refined scattering equation, where the "refinement" is the number of negative helicity particles, k. The number of solutions for the refined scattering equation at tree level was conjectured to have the pattern of Eulerian numbers $E(n-3, k-2)$ where n is the number of external particles [8]. In this project, we will prove this conjecture by solving for the recursion relation for the number of solutions of the refined scattering equation. This is done mathematically by using the method of dominance balance.

Despite an earlier proposal for an extension [9], it has remained a problem since. Later, Cachazo, He, and Yuan found a tree-level formula in arbitrary dimension [10]. It is a formula over the support of some remarkably simple constraints known as the scattering equations. The loop level of the CHY formula was extended in [11], where the formula is calculated on a nodal sphere on the support of the loop scattering equations. The number of solutions to the loop scattering equation were shown in [12]. This project shows that the number of solutions for the one loop CHY is what obtained in [12] and [13] in more detail using the method of dominance balance.

The progress on CHY inspired new works on the one-loop amplitude. In [14], by using a twistor theory in six dimensions, a formula was found for one loop. This formula is supported on constraints known as the loop polarized scattering equations, and the number of solutions is still unknown for these constraints. This project presents some work in progress on finding the number of solutions of the loop polarized scattering equations.

Since the number of solutions of the constraints in one loop twistor formula are still unknown, the aim of this project is to find them by establishing a recursion relation for the number of solutions in the soft limit using the method of dominance balance. This requires reviewing the mathematical and physical concepts of the twistor theory and its implication of calculating the scattering amplitude. We start by proving for the tree level RSVW formula that the number of solutions to its constraints are the Eulerian numbers. Then, to prepare for one loop calculation, we reproduce a proof of the number of solutions for the one loop scattering equations in[12] and [13], and present another proof using the method of dominance balance. Finally, after knowing what to expect of one loop level, we begin to develop a proof for the number of solutions of the one loop twistor formula.

The structure of this project is arranged as follows. In chapter 2 of this project, the difficulty of calculating Yang-Mills amplitude and the spinor-helicity formalism will be reviewed to lay the foundation for the next chapters. Then, in chapter 3, a brief introduction to the twistor theory and the construction of the massless fields on the twistor space are presented. The RSVW formula is presented in chapter 4, and the proof of the conjecture that the number of solutions to the constraints of RSVW formula are the Eulerian numbers will be presented in chapter 5.

The next two sections constitute the second half of the project, which is still work in progress, presenting the proofs for the loop-level number of solutions. In chapter 6, the CHY formula and its one-loop extension will be briefly illustrated, followed by the calculation that shows the recursion relation of the number of solutions in this theory. The last chapter (7) will provide some background on the 6d twistor amplitude and the loop polarized scattering equations. We present an attempt to find the recursion relation for the number of solutions. We make some remarks on the unexpected results for the work in progress.

Chapter 2

Amplitudes

The scattering processes that this project is concerned about are the scattering of gluons in super Yang-Mills theory, where the external particles are all gluons. The scattering amplitudes of gluons are not, however, directly measured in experiments. [1] Although the gluon scattering amplitudes will not be used in calculating real-experiments cross sections, it could show us the nature of Yang-Mills theory and tell us more of the structure that we have not previously seen in the non-perturbative level.

This section will start from introducing the Feynman diagrams approach of the gluons amplitude and complications that arise from the approach. The spinor-helicity formalism will allow us to simplify the calculation of the amplitude and show us that some of the amplitude can be determined by the helicity of the external particles. This section follows closely the first chapter of Elvang and Huang [2], with some additional comments from Schwartz's [1] and Srednicki's [15] textbooks.

2.1 Feynman Diagrams

The Yang-Mills Lagrangian in a mostly plus signature is given by

$$\mathcal{L} = -\frac{1}{4}\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu}\,, \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{ig}{\sqrt{2}}[A_\mu, A_\nu]$ and the gauge fields are in the adjoint representation of $SU(3)$, $A_\mu = A^a_\mu T^a$, with a running from 1 to 8. The normalization of the generators are given by $\mathrm{Tr}(T^aT^b) = \delta^{ab}$ and they satisfied the Lie algebra $[T^a, T^b] = if^{abc}T^c$. The standard method in calculating the scattering amplitudes is to use the Feynman diagrams approach. The diagrams represent the interaction of the particles in the theory, where each components such as the external lines, vertices, and the propagator can be extracted by fixing the gauge redundancy and read-off from the Lagrangian. A choice of gauge that leaves the Feynman rule simple and friendly for calculating the amplitudes is

[1] They are observed as jets of hadrons.

the Gervais-Neveu gauge. In this gauge, the gauge fixing term is

$$\mathcal{L}_{gf} = -\frac{1}{2}\,\mathrm{Tr}(H^{\mu}_{\mu})^2\,, \tag{2.2}$$

where $H_{\mu\nu} = \partial_\mu A_\nu - \frac{ig}{\sqrt{2}} A_\mu A_\nu$. With this choice of $H_{\mu\nu}$, the field strength tensor is observed to be the anti-symmetric part of $H_{\mu\nu}$:

$$F_{\mu\nu} = H_{\mu\nu} - H_{\nu\mu}\,. \tag{2.3}$$

Adding the gauge fixing term to the Yang-Mills Lagrangian yields

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\,\mathrm{Tr}(H^{\mu}_{\mu})^2 \\ &= -\frac{1}{2}\,\mathrm{Tr}\left(H^{\mu\nu}H_{\mu\nu} - H^{\mu\nu}H_{\nu\mu} + H^{\mu\mu}H_{\nu\nu}\right). \end{aligned} \tag{2.4}$$

Consider the terms with two derivatives from the second and the third term, one can do integration-by-parts to cancel them:

$$\begin{aligned}&-\mathrm{Tr}\left(\partial^\mu A^\nu \partial_\nu A_\mu - \partial^\mu A_\mu \partial^\nu A_\nu\right) \\ &\qquad = -\,\mathrm{Tr}\left(\partial_\nu(\partial^\mu A^\nu A_\mu) - A_\mu \partial^\mu \partial_\nu A^\nu - \partial^\mu(A_\mu \partial^\nu A^\nu) + A_\mu \partial^\mu \partial^\nu A_\nu\right) = 0\,,\end{aligned}$$

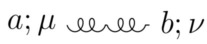
in which the total derivative term vanishes at the boundary.
The terms with no derivative from the second and third term cancel by using the cyclic property of the trace, and for the term with one derivative, we integrate-by-parts the last term:

$$\begin{aligned}&\mathrm{Tr}\left(\partial^\mu A^\nu A_\mu A_\nu - \partial^\mu A^\nu A_\nu A_\mu + \partial^\mu A_\mu A^\nu A_\nu\right) \\ &\quad = \mathrm{Tr}\left(\partial^\mu A^\nu A_\mu A_\nu - \partial^\mu A^\nu A_\nu A_\mu - A_\mu \partial^\mu (A^\nu A_\nu)\right) \\ &\quad = \mathrm{Tr}\left(\partial^\mu A^\nu A_\mu A_\nu - \partial^\mu A^\nu A_\nu A_\mu - A_\mu \partial^\mu (A^\nu) A_\nu) - A_\mu A^\nu \partial^\mu (A_\nu)\right) \\ &\quad = -2\,\mathrm{Tr}\left(\partial^\mu A^\nu A_\nu A_\mu\right).\end{aligned}$$

Combining all of them will find that the gauge fixed Lagrangian, ignoring the ghosts, is

$$\mathcal{L} = \mathrm{Tr}\left(-\frac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu - i\sqrt{2}g \partial^\mu A^\nu A_\nu A_\mu + \frac{1}{4} g^2 A^\mu A^\nu A_\mu A_\nu\right). \tag{2.5}$$

The propagator can be obtained from the Fourier transform of the kinetic term of the Lagrangian, taking the form of



$$= \frac{g^{\mu\nu}}{p^2 + i\epsilon}\delta^{ab}\,, \tag{2.6}$$

while the vertices from the interaction terms will involve the color factors f^{abc} and $f^{abe}f^{ecd}$ plus their permutations. We can simplify this by separating the color factors out and deal only with the kinematic structures. Using the identity

$$if^{abc} = \mathrm{Tr}([T^a, T^b]T^c) = \mathrm{Tr}(T^aT^bT^c) - \mathrm{Tr}(T^bT^aT^c)\,, \tag{2.7}$$

and the Fierz identity

$$\sum_a (T^a)_i^{\ j} (T^a)_k^{\ l} = \delta_i^l \delta_k^j - \frac{1}{N}\delta_i^j \delta_k^l\,, \tag{2.8}$$

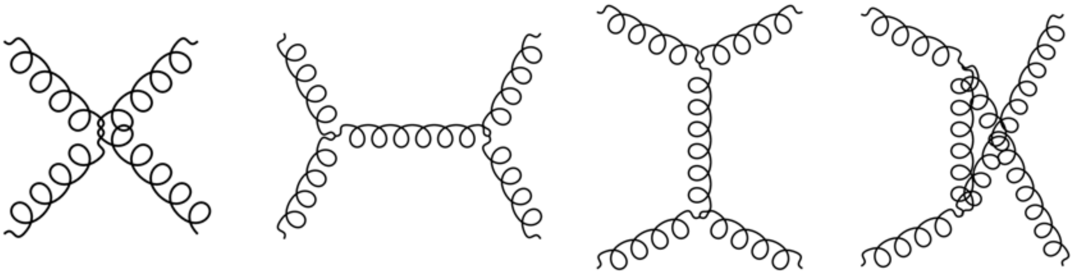
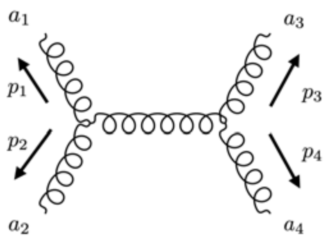


Figure 2.1: The four-point vertex, s-channel, t-channel, and u-channel.

one can re-write those products of generator-traces in the amplitude. As an illustration, the four external particle scattering amplitude consists of four main channel: the four-point vertex, and the s, t, and u channel. and these can be translate into equation by using the Feynman rule. However, if we look at one of them, the s-channel, which has two incoming particles labeled 1 and 2, and two outgoing particles labeled 3 and 4, the scattering amplitude is of the form



$$\sim \frac{g^2}{(-p_1-p_2)^2} f^{a_1a_2e} f^{ea_3a_4}(...) \tag{2.9}$$

where (...) will be determined from the vertex, which we will explicitly show shortly in this chapter. If we look at the colour factors for the s-channel, using the identities in (2.7) and (2.8), it can be re-written as

$$\begin{aligned} f^{a_1a_2e}f^{ea_3a_4} &= -\operatorname{Tr}([T^{a_1},T^{a_2}]T^e)\operatorname{Tr}([T^{a_3},T^{a_4}]T^e) \\ &= -\operatorname{Tr}([T^{a_1},T^{a_2}][T^{a_3},T^{a_4}]) + \frac{1}{N}\operatorname{Tr}([T^{a_1},T^{a_2}])\operatorname{Tr}([T^{a_3},T^{a_4}]) \\ &= -\operatorname{Tr}([T^{a_1},T^{a_2}][T^{a_3},T^{a_4}]) \\ &= -\operatorname{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) + \operatorname{Tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3}) \\ &\quad -\operatorname{Tr}(T^{a_1}T^{a_3}T^{a_4}T^{a_2}) + \operatorname{Tr}(T^{a_1}T^{a_4}T^{a_3}T^{a_2}). \end{aligned} \tag{2.10}$$

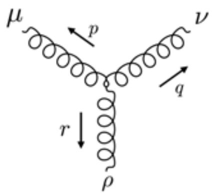
where at the second line, the product of two traces cancels using the cyclic property. The resulting object are the permutation of the generators with one generator fixed, here it is the permutation of (234). This mean that colour factors can be rewritten in terms of sum of the trace of permutation of generators. We can pull these traces out of each amplitudes and this leave us with the partial amplitude that have no colour factors. For 4-points, as an example, this is expressed as [2]

$$\mathbf{A}_4^{\text{tree}} = g^2\Big(A_4[1234]\,\text{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) + \text{permutations of }(234)\Big). \tag{2.11}$$

The ordering of the external particles in the partial amplitudes are fixed up to the trace factor, so they are also called the colour-ordered amplitudes. For n particles, we can write

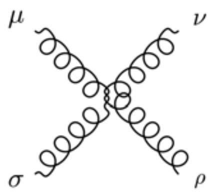
$$\mathbf{A}_n^{\text{tree}} = g^{n-2}\sum_{\text{perms}\,\sigma} A_n[1\sigma(2...n)]\,\text{Tr}\left(T^{a_1}T^{\sigma(a_2}...T^{a_n)}\right) \tag{2.12}$$

with the coupling constants are pulled out so that they will not be contained in the partial amplitudes. Thus, in the Gervais-Neveu gauge, the vertex rules for the 3-gluons and 4-gluons colour-ordered amplitudes are:



$$i\mathbf{V}_{\mu\nu\rho}(p,q,r) = -i\sqrt{2}[g^{\mu\nu}p_\rho + g^{\nu\rho}q_\mu + g^{\rho\mu}r_\nu]\,, \tag{2.13}$$

and



$$i\mathbf{V}_{\mu\nu\rho\sigma} = -i[g_{\mu\rho}g_{\nu\sigma}]\,. \tag{2.14}$$

After having the expression of the colour-ordered vertices, the amplitude can be calculate by just contracting the vertex with the polarization vectors ϵ^μ, and for more complicated amplitudes the usual rule of gluing vertices to internal propagator can be used.
For example, the triple vertex gives the amplitude

$$\begin{aligned} A_3[123] &= \epsilon_{1\mu}\epsilon_{2\nu}\epsilon_{3\rho}\mathbf{V}_{\mu\nu\rho}(p,q,r) \\ &= -\sqrt{2}\Big(\epsilon_1\cdot\epsilon_2\epsilon_3\cdot p + \epsilon_2\cdot\epsilon_3\epsilon_1\cdot q + \epsilon_2\cdot\epsilon_3\epsilon_1\cdot r\Big). \end{aligned} \tag{2.15}$$

The 4-point amplitudes is slightly more complicate since there are four diagrams: the 4-gluons vertex and the s, t, u channels. The 4-gluons vertex is directly obtained by contracting the polarization vectors, giving $\epsilon_2\cdot\epsilon_3\epsilon_1\cdot\epsilon_4$. However, the s-channel is not as simple as the 4-gluons vertex [3]

$$\begin{aligned} A_s[12\to 34] = \frac{2}{s}&\Big(\epsilon_1\cdot\epsilon_2\epsilon_q\cdot p_1 + \epsilon_q\cdot\epsilon_2\epsilon_1\cdot p_2 + \epsilon_q\cdot\epsilon_1\epsilon_2\cdot q\Big) \\ &\times\Big(\epsilon_3^*\cdot\epsilon_4^*\epsilon_q\cdot p_3 + \epsilon_q\cdot\epsilon_4^*\epsilon_3^*\cdot p_4 + \epsilon_q\cdot\epsilon_3^*\epsilon_4^*\cdot q\Big)\,, \end{aligned} \tag{2.16}$$

where $q = p_1 + p_2$, $s = -q^2 = -(p_1+p_2)^2$ and ϵ_q is the polarization vector of the propagating particle. This is not a trivial calculation and one can slightly simplify the

[2] The amplitude in this convention is written as $A[1234]$ and not $A[12\to 34]$ since we take all the particle to be outgoing for the convenience of expressing the identity. This exchange of incoming and outgoing particles will be explained in details in the next subsection.

[3] Here, the amplitude is written as the usual two incoming and two outgoing s channel.

structure by specifying the polarizations, but there are still a lot of terms for only this specific colour-ordered s-channel. After completing all colour-ordering of the s-channel, we have two more channels to go. In [1], the resulting cross section after all the calculation is remarkably simple and given by

$$\frac{1}{256}\sum_{\substack{\text{pols.}\\ \text{colours}}} |\mathbf{A}_4|^2 = g^4\frac{9}{2}\Big(3 - \frac{tu}{s^2} - \frac{su}{t^2} - \frac{st}{u^2}\Big)\,, \tag{2.17}$$

where $t = -(p_1 - p_3)^2$ and $u = -(p_1 - p_4)^2$.

With a lot of time and patience, the 4-point amplitude could be calculated. However, the number of terms grows quickly as the number of external particles are increased. Just by having one more external particle, there are 25 diagrams for the five points amplitude, containing 10000 terms, but most of the terms vanish due to the dot products of the polarization vectors. Since most of the terms are zero and the number of terms grow rapidly, and the resulting cross section looks suspiciously simple, maybe this method of Feynman diagram might not be the best way. A new method in calculating the amplitude was then introduced in the framework of the spinor-helicity variable.

2.2 Spinor-Helicity

The spinor-helicity formalism help us to organize the amplitude by the spin of the external particles by changing the basis of our calculation into the helicity basis as we are looking at gluons which are massless.[4] A strong suggestion that we need to consider working with the helicity or spin is that they are the main contributor of the vanishing of the terms in the amplitude. To see how the the "spinors" and "helicity" works, first, we would need to look at the Dirac spinors. The Lagrangian for the spinors is given by

$$\mathcal{L} = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi \tag{2.18}$$

where $\bar{\Psi} = \Psi^*\gamma^0$, and the field equations for this Lagrangian are the Dirac equations

$$(-i\gamma^\mu\partial_\mu + m)\Psi = 0 \tag{2.19}$$

by doing integration by parts, we obtain the field equation for the conjugate $\bar{\Psi}$

$$\bar{\Psi}(-i\overleftarrow{\partial}_\mu\gamma^\mu + m) = 0 \tag{2.20}$$

where the left arrow indicating that the partial derivative acts to its left. Acting on the Dirac equation with $(-i\gamma^\mu\partial_\mu + m)$, or take a "square", we get the Klein-Gordon equation

$$(-\partial^2 + m^2)\Psi = 0. \tag{2.21}$$

The solution to the Klein-Gordon is a wave solution, given by

$$\Psi(x) \sim u(p)e^{ip\cdot x} + v(p)e^{-ip\cdot x}, \tag{2.22}$$

[4]This helicity basis also works with other high energy fermions, as they also behave as if they are massless at high energy.

and we find the conditions that make the plane wave solution satisfies the Dirac equation by substituting the plane wave into the Dirac equation[[1]]. This give two conditions:

$$(\gamma^\mu p_\mu + m)u(p) = 0 \quad \text{and} \quad (-\gamma^\mu p_\mu + m)v(p) = 0. \tag{2.23}$$

Solving the first condition in the rest frame by letting $p = (m, 0, 0, 0)$ will yield two independent solutions for each equations, which are the two spin states:

$$u_s(p) = \begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix} \quad \text{and} \quad v_s(p) = \begin{pmatrix} \eta_s \\ -\eta_s \end{pmatrix}. \tag{2.24}$$

with $s = 1, 2$, the spin-up and spin-down. In general frame, they are

$$u_s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\xi_s \\ \sqrt{p \cdot \bar{\sigma}}\xi_s \end{pmatrix} \quad \text{and} \quad v_s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\eta_s \\ -\sqrt{p \cdot \bar{\sigma}}\eta_s \end{pmatrix}. \tag{2.25}$$

and for the conjugate $\bar{\Psi}$ we can have a similar result, which are the Dirac conjugates

$$\bar{u}_s(p) = u_s^\dagger(p)\gamma_0 \quad \text{and} \quad \bar{v}_s(p) = v_s^\dagger(p)\gamma_0. \tag{2.26}$$

where $\bar{u}_s(p)u_{s'}(p) = 2m\delta_{ss'}$ and $\bar{v}_s(p)v_{s'}(p) = -2m\delta_{ss'}$.
The general solution for the free field expansion of Ψ and $\bar{\Psi}$ is given by

$$\begin{aligned} \Psi(x) &= \sum_{s=\pm} \int \frac{d^3p}{(2\pi)^3 2E_p} \left[b_s(p)u_s(p)e^{ip\cdot x} + d_s^\dagger(p)v_s(p)e^{-ip\cdot x} \right] \\ \bar{\Psi}(x) &= \sum_{s=\pm} \int \frac{d^3p}{(2\pi)^3 2E_p} \left[b_s(p)^\dagger \bar{u}_s(p)e^{-ip\cdot x} + d_s(p)\bar{v}_s(p)e^{ip\cdot x} \right], \end{aligned} \tag{2.27}$$

and by quantizing, the operators $b_\pm(p), d_\pm(p)$ and $b_\pm^\dagger(p), d_\pm^\dagger(p)$ become fermionic annihilation and creation operators. Then the annihilation operator is used to define the vacuum state such that $b_\pm(p)|0\rangle = d_\pm(p)|0\rangle = 0$. This leads to the Feynman rules for the external fermions:

$$\begin{aligned} \text{incoming fermion} = u_s(p) &\qquad \text{outgoing fermion} = \bar{u}_s(p) \\ \text{incoming anti-fermion} = v_s(p) &\qquad \text{outgoing anti-fermion} = \bar{v}_s(p). \end{aligned} \tag{2.28}$$

For the massless particles, which we are interested in, the $\pm$ denotes the helicity of the particle, as the solution to the free equations of motion are the eigenstate of the helicity operator: $\frac{\vec{S}\cdot\vec{p}}{|\vec{p}|}\Psi = \pm\Psi$. The two helicities (negative and positive) can be viewed as left-handed and right-handed, as for the one with positive helicity, its spin and momentum are in the same direction (following the right-hand rule). These two spinors transform and live independently. These two helicity eigenstates are in the irreducible representations of the Lorentz group i.e. the two SU(2)s from looking at the double cover of the Lorentz group. We can separate the two helicity spinors by giving them different indices: positive helicity/right-handed with the dotted index and negative helicity/left-handed with the undotted index.

In this framework of the spinors-helicity eigenstates, we can rewrite the momentum by the contracting the four-momentum to the gamma matrices and the resulting object is a matrix:

$$\gamma^\mu p_\mu = \begin{pmatrix} 0 & p_\mu(\sigma^\mu)_{\alpha\dot{\beta}} \\ p_\mu(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} & 0 \end{pmatrix} \tag{2.29}$$

$$= \begin{pmatrix} 0 & p_{\alpha\dot{\beta}} \\ p^{\dot{\alpha}\beta} & 0 \end{pmatrix}, \tag{2.30}$$

with

$$p^{\dot{\alpha}\beta} = \begin{pmatrix} -p^0 - p^3 & -p^1 + ip^2 \\ p^1 - ip^2 & -p^0 + p^3 \end{pmatrix}, \tag{2.31}$$

where $\det p = -p^\mu p_\mu = m^2$. These $p_{\alpha\dot{\beta}}$ and $p_{\dot{\alpha}\beta}$ are called the momentum bi-spinors as they live in the $(\frac{1}{2}, \frac{1}{2})$ representation.

For massless particles, the Dirac equations became

$$\gamma^\mu p_\mu v_\pm(p) = 0 \quad \text{and} \quad \bar{u}_\pm(p)\gamma^\mu p_\mu = 0\,, \tag{2.32}$$

and we focus on $\bar{u}_\pm$ and $v_\pm$ as they are the outgoing fermions and antifermions, and the incoming particles can be expressed as their conjugates. Due to the crossing symmetry, that is, incoming positive helicity (anti-)fermion is equivalent to outgoing negative helicity (anti-)fermion and vice versa, the u's and v's are related as $u_\pm = v_\mp$ and $\bar{u}_\pm = \bar{v}_\mp$.

As mentioned above that the two helicity bases live in two different representation – the left-handed and right-handed – the two solutions to the massless Dirac equations can be written as

$$v_+(p) = \begin{pmatrix} \tilde{\lambda}_{\dot{\alpha}} \\ 0 \end{pmatrix} \quad \text{and} \quad v_-(p) = \begin{pmatrix} 0 \\ \lambda^\alpha \end{pmatrix}, \tag{2.33}$$

and

$$\bar{u}_+(p) = \begin{pmatrix} \tilde{\lambda}^{\dot{\alpha}} & 0 \end{pmatrix} \quad \text{and} \quad \bar{u}_-(p) = \begin{pmatrix} 0 & \lambda_\alpha \end{pmatrix}, \tag{2.34}$$

where the λ and $\tilde{\lambda}$ are 2-component commuting spinors wielding the undotted and dotted index of the 2 $SU(2)$s. Substituting these back into the Dirac equation will show that these spinors satisfy the Weyl equation:

$$p^{\dot{\alpha}\beta}\tilde{\lambda}_{\dot{\alpha}} = 0, \quad \lambda^\alpha p_{\alpha\dot{\beta}} = 0, \quad \tilde{\lambda}^{\dot{\beta}} p_{\alpha\dot{\beta}} = 0, \quad p^{\dot{\alpha}\beta}\lambda_\beta = 0. \tag{2.35}$$

Using the spin-sum completeness relation for massless spinors, one can show that

$$-\gamma^\mu p_\mu = u_-\bar{u}_- + u_+\bar{u}_+ = \lambda_\alpha\tilde{\lambda}_{\dot{\alpha}} + \lambda^\alpha\tilde{\lambda}^{\dot{\alpha}} \tag{2.36}$$

$$= p_{\alpha\dot{\alpha}} + p^{\dot{\alpha}\alpha}\,, \tag{2.37}$$

where the 2×2 matrices $p_{\alpha\dot{\alpha}}$ and $p^{\dot{\alpha}\alpha}$ lies in the off-diagonal block of the 4×4 matrix $\gamma^\mu p_\mu$.

This decomposition of a massless momentum into two spinors is not a coincidence, it can be done due to the fact that the norm of a vector v^μ mapped into the $SL(2,\mathbb{C})$ is given by the determinant $\det(v^{\alpha\dot{\alpha}})$. We can see that the determinant vanishes for any null vector, and if the determinant of a matrix is zero then its rank must be lower than its dimension (which is two). Therefore, any null vector can be written as

$$v^{\alpha\dot{\alpha}} = a^\alpha \tilde{a}^{\dot{\alpha}} \,, \tag{2.38}$$

for some spinors a^α and $\tilde{a}^{\dot{\alpha}}$.
The raising and lowering of the indices can be done by using the two-dimensional Levi-Cevita symbol,

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon_{\dot{\alpha}\dot{\beta}} \,, \tag{2.39}$$

where its inverse is $\epsilon^{\alpha\gamma}\epsilon_{\beta\gamma} = \delta^\alpha_\beta$ and $\epsilon^{\alpha\beta}\epsilon_{\alpha\beta} = 2$. Its anti-symmetry requires us to fix a convention for raising and lowering the index. We use the convention of "lower to the right, raise to the left" or what some might called the "Northwest - Southeast" convention:

$$a_\alpha = a^\beta \epsilon_{\beta\alpha} \quad \text{and} \quad a^\alpha = \epsilon^{\alpha\beta} a_\beta \,. \tag{2.40}$$

The dual vector can then be obtained by

$$v_{\alpha\dot{\alpha}} = v^{\beta\dot{\beta}} \epsilon_{\beta\alpha} \epsilon_{\dot{\beta}\dot{\alpha}} \,, \tag{2.41}$$

and the inner product of the vectors is

$$v^{\alpha\dot{\alpha}} v_{\alpha\dot{\alpha}} = \det(v_{\alpha\dot{\alpha}}) \,. \tag{2.42}$$

The inner products of the spinors are denoted by the angle and square brackets:

$$\langle ab \rangle = a^\alpha b_\alpha = a^\alpha b^\beta \epsilon_{\alpha\beta}, \quad \text{and} \quad [\tilde{a}\tilde{b}] = \tilde{a}^{\dot{\alpha}} \tilde{b}_{\dot{\alpha}} = \tilde{a}^{\dot{\alpha}} \tilde{b}^{\dot{\beta}} \epsilon_{\dot{\alpha}\dot{\beta}} \,, \tag{2.43}$$

where $\langle ab \rangle = -\langle ba \rangle$ and $\langle aa \rangle = 0 = [\tilde{a}\tilde{a}]$. [5] In this notation, the inner product of two null vectors $v_{\text{null}} = \nu^\alpha \tilde{\nu}^{\dot{\alpha}}$ and $w_{\text{null}} = \omega^\alpha \tilde{\omega}^{\dot{\alpha}}$ is given by

$$v_{\text{null}} \cdot w_{\text{null}} = \frac{1}{2} \nu^\alpha \tilde{\nu}^{\dot{\alpha}} \omega_\alpha \tilde{\omega}_{\dot{\alpha}} = \frac{1}{2} \langle \nu\omega \rangle [\tilde{\nu}\tilde{\omega}] . \tag{2.44}$$

2.3 Calculating the Amplitude

The last ingredient we need before calculating the amplitude is the polarization vectors. The polarization vectors satisfies the condition that $\epsilon_\pm \cdot \epsilon_\pm = 0$, $\epsilon_\pm \cdot \epsilon_\mp = -1$, and $\epsilon_\pm \cdot p = 0$. Following the spinor-helicity formalism, the polarization vector can be decompose into two spinors. The last condition $\epsilon_\pm \cdot p = 0$ suggests that for $p^{\alpha\dot{\alpha}} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}$ and a reference momentum $r^{\alpha\dot{\alpha}} = \zeta^\alpha \tilde{\zeta}^{\dot{\alpha}}$

$$[\epsilon_p^-(r)]^{\alpha\dot{\alpha}} = \sqrt{2} \frac{\lambda^\alpha \tilde{\zeta}^{\dot{\alpha}}}{[\tilde{\lambda}\tilde{\zeta}]} \quad \text{and} \quad [\epsilon_p^+(r)]^{\alpha\dot{\alpha}} = \sqrt{2} \frac{\zeta^\alpha \tilde{\lambda}^{\dot{\alpha}}}{\langle \lambda\zeta \rangle}, \tag{2.45}$$

[5] This reflects that they are commuting spinors and not the Grassmannian spinors since the spinors that satisfies $\psi^\alpha \chi^\beta = -\chi^\beta \psi^\alpha$ will have a symmetric inner product $\langle \psi\chi \rangle = \langle \psi\chi \rangle$.

where the denominators and the $\sqrt{2}$ are for the normalization to satisfies $\epsilon_\pm \cdot \epsilon_\mp = -1$.[6] The subscript denote the momentum the this polarization is associated, and the argument denote the reference momentum.

For our further convenience, we might as well calculate the dot products between different polarization vectors:

$$\epsilon_1^-(i) \cdot \epsilon_2^-(j) = \frac{\langle 12\rangle [ji]}{[1i][2j]}\,, \quad \epsilon_1^+(i) \cdot \epsilon_2^+(j) = \frac{\langle ij\rangle [21]}{\langle i1\rangle\langle j2\rangle}\,, \quad \epsilon_1^-(i) \cdot \epsilon_2^+(j) = \frac{\langle 1j\rangle [2i]}{[1i]\langle 2j\rangle}\,, \tag{2.46}$$

$$\epsilon_1^-(i) \cdot p_2 = \frac{1}{\sqrt{2}}\frac{\langle 12\rangle [2i]}{[1i]} \quad \text{and} \quad \epsilon_1^+(i) \cdot p_2 = \frac{1}{\sqrt{2}}\frac{[12]\langle 2i\rangle}{\langle i1\rangle} \tag{2.47}$$

Another useful identity of manipulating the spinor helicity variables is called the Schouten identity. This follows from the fact that in $2d$, there cannot be more than 2 independent vectors. So we can always write a vector into a linear combination of the others:

$$|k\rangle = a|i\rangle + b|j\rangle. \tag{2.48}$$

Since we know that the product $\langle ii\rangle$ vanishes, we can find a and b by dotting in $\langle j|$ and $\langle i|$, respectively. The result is

$$|k\rangle = \frac{\langle jk\rangle}{\langle ji\rangle}|i\rangle + \frac{\langle ik\rangle}{\langle ij\rangle}|j\rangle, \tag{2.49}$$

and by dotting in another spinor we have a beautiful Schouten identity:

$$\langle ri\rangle\langle jk\rangle + \langle rj\rangle\langle ki\rangle + \langle rk\rangle\langle ij\rangle = 0. \tag{2.50}$$

After we have laid down the foundation for the spinor-helicity formalism, we can go back to the s-channel amplitude from the previous section using the spinor-helicity variable. Recall the s-channel amplitude,

$$A_s[12\to 34] = \frac{2}{s}\Big(\epsilon_1\cdot\epsilon_2\epsilon_q\cdot p_1 + \epsilon_q\cdot\epsilon_2\epsilon_1\cdot p_2 + \epsilon_q\cdot\epsilon_1\epsilon_2\cdot q\Big)\times\Big(\epsilon_3^*\cdot\epsilon_4^*\epsilon_q\cdot p_3 + \epsilon_q\cdot\epsilon_4^*\epsilon_3^*\cdot p_4 + \epsilon_q\cdot\epsilon_3^*\epsilon_4^*\cdot q\Big), \tag{2.51}$$

the amplitude in any channel depends on the products of polarization, so we need to specify the helicity of each particle in order to calculate the amplitude. For the amplitude with all positive helicity external particles, by selecting the reference momentum to be some arbitrary momentum r for all external particles, we have that

$$\epsilon_1^+(r) \cdot \epsilon_2^+(r) = \frac{\langle \zeta\zeta\rangle [21]}{\langle \zeta 1\rangle\langle \zeta 2\rangle} = 0\,, \tag{2.52}$$

and therefore

$$A[1^+2^+3^+4^+] = 0\,, \tag{2.53}$$

for whatever channel we are looking at, and the same argument goes for the all minus helicity:

$$A[1^-2^-3^-4^-] = 0\,. \tag{2.54}$$

[6]The condition $p\cdot\epsilon = 0$ came from the field equation $\partial A = 0$ while the normalization is conventional.

If one of the external particle has a different helicity, let's say 1st has negative helicity. We can select the reference momentum for the first particle to be some momentum k, and for the positive helicity particles we choose them to be p_1. All the products of the polarization vectors with the same helicity vanish, and since

$$\epsilon_1^-(k) \cdot \epsilon_2^+(p_1) = \frac{\langle 11 \rangle [2k]}{[1k]\langle 21 \rangle} = 0\,, \tag{2.55}$$

the amplitude vanishes

$$A[1^-2^+3^+4^+] = 0\,, \tag{2.56}$$

and the same goes for the opposite helicity:

$$A[1^+2^-3^-4^-] = 0\,. \tag{2.57}$$

So now we are left with $A[--++]$ and its permutations. We can select the reference momenta to be p_3 for particle 1 and 2 and p_2 for particle 3 and 4. (Diagram of s-channel, t-channel, and 4-gluons) The four-gluons vertex is given by

$$\sim (\epsilon_1 \cdot \epsilon_3)(\epsilon_2 \cdot \epsilon_4)\,,$$

and this vanishes since $\epsilon_1^-(p_3) \cdot \epsilon_3^+(p_2) = 0$. For the t-channel diagram, the $23q$ vertex is of the form

$$\mathbf{V}(2,3,q) \sim \epsilon_2^- \cdot \epsilon_3^+ \epsilon_q \cdot p_2 + \epsilon_q \cdot \epsilon_3^+ \epsilon_2^- \cdot p_3 + \epsilon_q \cdot \epsilon_2^- \epsilon_3^+ \cdot q\,, \tag{2.58}$$

where ϵ_q is just a placeholder for the internal polarization. It can be seen that $\epsilon_2^- \cdot \epsilon_3^+ = 0$ so the first term vanish, also since $q = -p_2 - p_3$ and the reference momenta of particle 2 is p_3 and vice versa, the products $q \cdot \epsilon_3^+$ and $q \cdot \epsilon_2^-$ vanish. Therefore, the only diagram contributing to $(--++)$ is the s-channel.
The amplitude for this channel can be calculate by gluing the two vertices together with the propagator. Here the internal polarizations are given by ϵ_q and $\epsilon_{q'}$, where they would be replace by the propagator as we glue them

$$\begin{aligned} A[--++] &= 2\Big(\epsilon_1^- \cdot \epsilon_2^- \epsilon_q \cdot p_1 + \epsilon_q \cdot \epsilon_2^- \epsilon_1 \cdot p_2 + \epsilon_q \cdot \epsilon_1^- \epsilon_2 \cdot q\Big) \\ &\quad \times \Big(\epsilon_3^+ \cdot \epsilon_4^+ \epsilon_q \cdot p_3 + \epsilon_{q'} \cdot \epsilon_4^+ \epsilon_3^+ \cdot p_4 + \epsilon_{q'} \cdot \epsilon_3^+ \epsilon_4^+ \cdot q'\Big)\,, \\ &= 2\Big(\epsilon_q \cdot \epsilon_2^- \epsilon_1^- \cdot p_2 + \epsilon_q \cdot \epsilon_1^- \epsilon_2^- \cdot q\Big) \times \Big(\epsilon_{q'} \cdot \epsilon_4^+ \epsilon_3^+ \cdot p_4 + \epsilon_{q'} \cdot \epsilon_3^+ \epsilon_4^+ \cdot q'\Big)\,. \end{aligned} \tag{2.59}$$

The first term of both vertices vanishes, and by gluing the vertices together we will take the internal polarizations to be the propagator: $\epsilon_q^\mu \epsilon_{q'}^\nu \to g^{\mu\nu}$:

$$\begin{aligned} A[--++] &= \frac{2}{s}\Big(\epsilon_4^+ \cdot \epsilon_2^- \epsilon_1^- \cdot p_2 \epsilon_3^+ \cdot p_4 + \epsilon_2^- \cdot \epsilon_3^+ \epsilon_1^- \cdot p_2 \epsilon_4^+ \cdot q' \\ &\qquad + \epsilon_4^+ \cdot \epsilon_1^- \epsilon_2^- \cdot q \epsilon_3^+ \cdot p_4 + \epsilon_3^+ \cdot \epsilon_1^- \epsilon_2^- \cdot q \epsilon_4^+ \cdot q'\Big) \\ &= \frac{2}{s}\Big((\epsilon_4^+ \cdot \epsilon_1^-)(\epsilon_2^- \cdot q)(\epsilon_3^+ \cdot p_4)\Big)\,, \end{aligned} \tag{2.60}$$

since $\epsilon_4^+(p_2) \cdot \epsilon_2^-(p_3) = 0 = \epsilon_2^-(p_3) \cdot \epsilon_3^+(p_2) = \epsilon_1^-(p_3) \cdot \epsilon_3^+(p_2)$. Using the formula we have written above, $q = -p_1 - p_2$, and $s = (-p_1 - p_2)^2 = \langle 12 \rangle [21]$

$$A[--++] = \frac{1}{s}\left(\frac{\langle 12 \rangle [43]}{[13]\langle 24 \rangle}\right)\left(-\frac{\langle 21 \rangle [13]}{[23]}\right)\left(\frac{[34]\langle 42 \rangle}{\langle 23 \rangle}\right) = \frac{\langle 12 \rangle [34]^2}{[21]\langle 23 \rangle [23]}. \tag{2.61}$$

Using the conservation of momentum, $\langle 12\rangle[21] = \langle 34\rangle[43]$,

$$\frac{\langle 12\rangle[34]^2}{[21]\langle 23\rangle[23]} = \frac{\langle 12\rangle^2[34]}{\langle 23\rangle\langle 43\rangle[23]}\,,$$

and using the conservation of momentum identity $\sum_j \langle ij\rangle[jk] = 0$ for $i = 1, k = 3$, we have that $\langle 12\rangle[23] + \langle 14\rangle[43] = 0$ so $\frac{[43]}{[32]} = \frac{\langle 12\rangle}{\langle 14\rangle}$, we can write the amplitude as

$$A[1^-2^-3^+4^+] = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,. \tag{2.62}$$

For other color-ordering that the negative helicity are adjacent, we can use the cyclic symmetry to yield

$$A[1^+2^-3^-4^+] = \frac{\langle 23\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,. \tag{2.63}$$

To obtain the amplitude for $A[1^-2^+3^-4^+]$, we must first consider an important identity called the $U(1)$-decoupling identity. This is by taking a generator to be proportional to the identity matrix, then the group correspond to a $U(N)$ rather than an $SU(N)$ with that identity being the generator of the extra $U(1)$, commuting with the other $SU(N)$ generators. Then if we write a tree level amplitude with one of the generator replaced by the identity matrix, the result must be zero:

$$\mathbf{A}_n^{\text{tree}}\Big|_{\text{U(1) decouples}} = 0 = g^{n-2}\sum_{\text{perms}\sigma} A_n[1\sigma(2...n)]\,\text{Tr}\left(\mathbb{1}T^{\sigma(a_2}...T^{a_n)}\right) \tag{2.64}$$

This identity is also reflected in the vanishing of the term with $1/N$ in calculating the Fierz identity of $f^{a_1a_2e}f^{ea_3a_4}$.
Using this identity with $T^{a_4} = \mathbb{1}$, the amplitude $A[1^-2^+3^-4^+]$ is

$$\begin{aligned} A[1^-2^+3^-4^+] &= -A[2^+3^-1^-4^+] - A[3^-1^-2^+4^+] \\ &= -\frac{\langle 13\rangle^4}{\langle 23\rangle\langle 31\rangle\langle 14\rangle\langle 42\rangle} - \frac{\langle 13\rangle^4}{\langle 31\rangle\langle 12\rangle\langle 24\rangle\langle 43\rangle} \\ &= \frac{\langle 13\rangle^4}{\langle 31\rangle\langle 24\rangle}\left[\frac{\langle 12\rangle\langle 43\rangle - \langle 14\rangle\langle 23\rangle}{\langle 14\rangle\langle 23\rangle\langle 12\rangle\langle 43\rangle}\right] \\ &= \frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,. \end{aligned} \tag{2.65}$$

where we have used the Schouten identity $\langle 12\rangle\langle 34\rangle + \langle 14\rangle\langle 23\rangle = -\langle 13\rangle\langle 42\rangle$.

The form of all amplitudes with two negative and two positive helicity particles have a similar feature, that we can write the amplitudes as the inner product of the two negative helicities on top of the cyclic terms. We can also write this amplitude in terms of the other helicity spinors:

$$A[1^-2^-3^+4^+] = \frac{[34]^4}{[12][23][34][41]}\,, \tag{2.66}$$

by using the momentum conservation relations. In fact, this is a special case of a formula called the Parke-Taylor formula for an amplitude with 2 opposite helicity external particles [3]. For the opposite helicity chosen to be negative, the Parke-Taylor formula for n external particles is

$$A_n[1^+ ... i^- ... j^- .... n^+] = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle ... \langle n1\rangle} . \tag{2.67}$$

We shall not prove nor derive this formula here but we can use a fundamental property of the amplitude, that is counting the little group scaling, to see that this formula holds.

2.4 Little Group Scaling and MHV Classification

A massless momentum can always be written in a frame that the z-axis is aligned with the direction of motion: $p^\mu = (E, 0, 0, E)$. We can see that this momentum is invariant under rotation in the $xy-$plane. This is called the little group scaling, as it is a smaller set of transformation that leaves the (on-shell) momentum invariant.
If we look at the mapping of the momentum into two spinors, we can see that the momentum is left invariant under the transformation

$$\lambda^\alpha \to t\lambda^\alpha \quad \text{and} \quad \tilde{\lambda}^{\dot{\alpha}} \to t^{-1}\tilde{\lambda}^{\dot{\alpha}} . \tag{2.68}$$

The little group transformation is now realized as a scaling of the spinors in the decomposition.
Under this little group scaling of massless particles, we can see that the scalars remain invariant, while the spinors with helicity $h = \pm\frac{1}{2}$ scale as t^{-2h}, and vectors such as the spin-1 polarization vectors scale as t^{-2h} for $h = \pm 1$. The scattering amplitudes are composed of these variables, so if we scale the i^{th} particle's variable under little group we know that we must have a factor of $t_i^{-2h_i}$. Thus, the amplitude of massless particles scales as

$$A_n[(\lambda_1, \tilde{\lambda}_1, h_1), ..., (t_i\lambda_i, t_i^{-1}\tilde{\lambda}_i, h_i), ...] = t_i^{-2h_i} A_n[(\lambda_1, \tilde{\lambda}_1, h_1), ..., (\lambda_i, \tilde{\lambda}_i, h_i), ...] . \tag{2.69}$$

Under the little group transformation of the scattering amplitude, we can look at the three particle amplitude as an example, and as we shall see, the little group scaling will fix the structure of the amplitudes. Let us suppose that the amplitude consists only of the angled brackets, a general form for the amplitude is

$$A_3(1^{h_1}2^{h_2}3^{h_3}) = c\langle 12\rangle^{x_{12}}\langle 13\rangle^{x_{13}}\langle 23\rangle^{x_{23}} . \tag{2.70}$$

The little group scaling fixes the exponents to satisfy

$$-2h_1 = x_{12} + x_{13}, \quad -2h_2 = x_{12} + x_{23}, \quad \text{and} \quad -2h_3 = x_{13} + x_{23} , \tag{2.71}$$

and this gives

$$x_{12} = h_3 - h_1 - h_2, \quad x_{13} = h_2 - h_1 - h_3, \quad \text{and} \quad x_{23} = h_1 - h_2 - h_3 . \tag{2.72}$$

The amplitude then depends on the helicities of the particles:

$$A_3(1^{h_1}2^{h_2}3^{h_3}) = c\langle 12\rangle^{h_3-h_1-h_2}\langle 13\rangle^{h_2-h_1-h_3}\langle 23\rangle^{h_1-h_2-h_3} . \tag{2.73}$$

This gives an expression for an amplitude up to an overall factor. We can see that for 2 negative helicity and 1 positive helicity, the amplitude is

$$A_3[1^-2^-3^+] = c\frac{\langle 12\rangle^3}{\langle 13\rangle\langle 23\rangle} = c\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\,, \tag{2.74}$$

matching the Parke-Taylor and what we have calculated before. However, if we assume that the amplitude must consists of all square brackets, we will instead have

$$A_3[1^-2^-3^+] = c\frac{[13]\,[23]}{[12]^3}\,. \tag{2.75}$$

This is a valid amplitude under the little group scaling, but if we look at the mass dimension of the amplitude (i.e. each spinors has mass dimension 1), the amplitude with angle bracket is a (mass)^1 and this one with square bracket is a (mass)$^{-1}$. The mass dimension reflects the term in the Lagrangian that it came from. So, the angle brackets with (mass)^1 is compatible with the fact that it is from the term $AA\partial A$, but the (mass)$^{-1}$ term said that it must be from $AA\frac{\partial}{\Box}A$. This term is not present in a local Lagrangian, so this amplitude of square bracket is unphysical and must be discard. The moral of the story is that we can determine the amplitude from the little group scaling and locality.

Another point that must be aware of is that the Yang-Mill coupling g is dimensionless, but for the $AA\frac{\partial}{\Box}A$ term, we must have a coupling g' with dimension (mass)^2 (such that the mass dimension is (mass)^4). This means that the amplitude that we have derived all have dimension (mass)^1. In fact, the amplitude for n external particle in four dimension must have mass dimension $4-n$.

This scaling will allow us to see why the Parke-Taylor formula works. An amplitude consists of vertices on the numerator and the propagators at the denominator. In Yang-Mills theory, we can only have trivalent graphs i.e. graphs built from three-point vertices. [7] Every time another external line is added, a vertex and a propagator must be added to the graph to keep it trivalent. So the number of vertex and propagator grows with n. Since the first vertex took three particles, number of vertex grows as $n-2$, and thus creating the first propagator at two vertices (four particles), the number of propagators then grow as $n-3$. The cubic vertices is of (mass)^1 and the propagator is (mass)^2, and the dimension of the amplitude is then

$$[A_n] \sim \frac{(\text{mass})^{n-2}}{(\text{mass}^2)^{n-3}} \sim (\text{mass})^{4-n}\,. \tag{2.76}$$

First, recall that the amplitudes for all-plus and all-minus helicity particles vanish due to the fact that the dot products of the polarization vectors are all zero. Next, an amplitude with all-except-one minus/plus helicity particles $A_n[1^\pm....i^\mp...n^\pm]$ also vanishes by choosing the reference momentum of the majority to be the momentum of the different one p_i, thus $\epsilon_j^\pm\cdot\epsilon_k^\pm = 0$ and $\epsilon_i^\pm\cdot\epsilon_k^\pm = 0$ for all j and k.

[7] As we have shown in the last section, the four-point vertex vanishes.

For the two flipped helicities $A_n[1^-2^-3^+...n^+]$, we can choose the reference momentum as $r_1 = r_2 = p_n$ and $r_3 = ... = r_n = p_1$, and this results in the vanishing of all dot products of polarization vectors except $\epsilon_2^- \cdot \epsilon_i^+$ for $i = 3, ..., n-1$. Schematically, this amplitude takes the form of

$$A_n[1^-2^-3^+...n^+] \sim \sum_{\text{diagrams}} \frac{\sum(\epsilon_2^- \cdot \epsilon_i^+)(\epsilon_j \cdot k_l)^{n-2}}{\prod P_I^2}\,, \tag{2.77}$$

where P_I^2 are the square of the propagators. The mass dimension of the numerator is $n-2$, agreeing with the dimensional analysis. Also, since tree level diagrams for $A_n[1^-2^-3^+...n^+]$ are all trivalent, these $n-2$ factors of $\epsilon_j \cdot k_l$ reflect the fact that all vertices are trivalent. At this stage, we can see that $A_n[1^-2^-3^+...n^+]$ is the first[8] non-vanishing amplitude. The amplitudes with two negative helicities and $n-2$ positive helicities are called the Maximally Helicity Violating (MHV) amplitude, where the one with two positive and $n-2$ negative are called the anti-MHV or $\overline{\text{MHV}}$.

The helicity violation can be seen if we flip some of the outgoing particle to be an incoming particle with opposite helicity. The two outgoing negative helicities can be viewed as two incoming positive helicities. The vanishing amplitudes with all plus/minus and with one plus/minus helicities in this crossing over became $1^-2^- \to 3^+...n^+$ and $1^-2^+ \to 3^+...n^+$, which could be seen right away that they violate the helicity conservation. The MHV amplitude can then be written as $1^+2^+ \to 3^+...n^+$. Since it is the next order and non-vanishing, we say this is the most we can violate the helicity conservation. We can also look at the other processes with two flipped helicities, for example, $A_n[1^+2^+3^-4^-...n^+]$, flipped to $1^-2^- \to 3^-4^-...n^+$, that they do not vanish. So, to conclude, these amplitudes all violate helicity conservation, but the MHV is the most we can violate.

The amplitude of the MHV are given by the Parke-Taylor formula, as given by (2.67),

$$A_n[1^+...i^-...j^-....n^+] = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle...\langle n1\rangle}\,, \tag{2.78}$$

and the anti-MHV is

$$A_n[1^-...i^+...j^+....n^-] = \frac{[ij]^4}{[12]\,[23]...[n1]}\,. \tag{2.79}$$

The Parke-Taylor formula scales properly under the little group scaling and has the correct mass dimension of $4-n$.

From supersymmetry, one can obtain the Parke-Taylor formula from the superamplitude (C.14) given in the appendix A

$$\mathcal{A}_n^{\text{MHV}} = \frac{\delta^8(\tilde{\mathcal{Q}})\delta^4(\sum_{i=1}^n \kappa_i^\alpha \tilde{\kappa}_i^{\dot{\alpha}})}{\langle 12\rangle\,\langle 23\rangle\,...\,\langle n1\rangle}\,,$$

[8] First in the sense that we ordered the amplitudes according to the number of negative helicity particles.

by applying derivatives with respect to the Grassmannian variables η_{iA} and η_{jA}. Here, $\tilde{\mathcal{Q}}$ is the total supercharge $\tilde{\mathcal{Q}}_A^\alpha := \sum_{i=1^n} \kappa_i^\alpha \eta_{iA}$, and the delta function is given in (C.12) as

$$\delta^{(8)}(\tilde{\mathcal{Q}}) = \frac{1}{2^4} \prod_{A=1}^{4} \sum_{i,j=1}^{n} \langle ij \rangle \, \eta_{iA} \eta_{jA} \, .$$

One may ask about the other amplitudes between MHV and anti-MHV. Certainly, the amplitudes with three negative helicities and $n-3$ positive helicities are non-vanishing, but they might not look so simple like the MHVs. These amplitudes are called the Next-to-MHV amplitudes or N^{k-2} MHV, with k being the number of negative helicity gluons. There are many different ways of studying the N^{k-2} MHV amplitudes, one of them is to look at the pole structure of the amplitude under some limit. These poles will correspond to the internal propagator which will factorize an amplitude into two, allowing us to write a recursion relation, known as the BCFW recursion relations. The MHV and NMHV amplitudes were studied using the twistor formalism in [5] and was found that the scattering amplitude can be generalized as an integral supported over curves of degree $d = k - 1$.

Chapter 3

Twistor Theory

Twistor theory aims to describe light rays in a compact way, by making a correspondence between the conformal structure of a light cone with a holomorphic Riemann sphere, with the polarization of the light rays encoded in the spinorial structure. In a simple picture, a point in twistor space corresponds to a null ray on Minkowski space, and, for the other way round, a point in Minkowski space corresponds to a line (or Riemann sphere) on the twistor space.

In this project, one of the modern application of twistor theory – in studying scattering amplitudes – will be explored. In particular, the formula given by Roiban, Spradlin, Volovich, and Witten that will be introduced in the next chapter, uses the twistor space as the framework in calculating the MHV and NMHV amplitudes. Therefore, some aspects of twistor theory essential to the RSVW formula will be covered in this chapter. We follow an introduction to the twistor theory by Adamo [16] to introduce the twistor space and how to describe the massless fields on the twistor space using the Penrose transform.

3.1 Twistor Space

The twistor formalism uses the complex structures to describe the conformal structure of spacetime, so it requires the complexification of the Minkowski space. The complexification is to allow the coordinates x^μ to take complex value instead of just real, and extend the metric $g_{\mu\nu}$ holomorphically (in a way that it does not depends on the conjugate variables). The complexified Minkowski space $\mathcal{M}_\mathbb{C}$ can later recover the signature of spacetime by consider a slice of $\mathcal{M}_\mathbb{C}$ by imposing the reality conditions.

The benefit of complexification is that it allows us to look at the complexified Lorentz group $SO(4,\mathbb{C})$ instead of the usual Lorentz with specified signature. The complexified Lorentz group is locally isomorphic to $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. This meant that a vector on $\mathcal{M}_\mathbb{C}$ can be mapped into the representation of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$, and can be described using a pair of $SL(2,\mathbb{C})$ index.

$$v^{\alpha\dot{\alpha}} := \frac{\sigma_a^{\alpha\dot{\alpha}}}{\sqrt{2}} v^a = \frac{1}{\sqrt{2}} \begin{pmatrix} v^0 + v^3 & v^1 - iv^2 \\ v^1 + iv^2 & v^0 - v^3 \end{pmatrix} . \tag{3.1}$$

Similar to the spinor-helicity construction, any tensor can be mapped into this representation by contracting with the Pauli matrices, resulting in an object with each vector index replaced with a pair of spinor indices. In this section, we normalized the representation of the vector in $SL(2,\mathbb{C})$ with a factor of $1/\sqrt{2}$, so that $v_{\text{null}} \cdot w_{\text{null}} = \langle \nu\omega \rangle [\tilde{\nu}\tilde{\omega}]$.

From this point, one can retrieve the $SU(2)$ spinors by selecting a slice of Lorentzian real Minkowski space inside of $\mathcal{M}_{\mathbb{C}}$. This is done by selecting all component of the coordinate (x^0, x^1, x^2, x^3) to be real, equivalent to having the matrix $x^{\alpha\dot{\alpha}}$ being Hermitian, with

$$x^{\alpha\dot{\alpha}\dagger} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{x}^0 + \bar{x}^3 & \bar{x}^1 - i\bar{x}^2 \\ \bar{x}^1 + i\bar{x}^2 & \bar{x}^0 - \bar{x}^3 \end{pmatrix} . \tag{3.2}$$

The Hermitian conjugate includes the transpose, so the positive and negative chirality parts are exchanged under the conjugation. Explicitly, the conjugation acts on each spinor as

$$\kappa^\alpha = (a, b) \to \bar{\kappa}^{\dot{\alpha}} = (\bar{a}, \bar{b}) \quad \text{and} \quad \tilde{\omega}^{\dot{\alpha}} = (c, d) \to \bar{\tilde{\omega}}^\alpha = (\bar{c}, \bar{d}) , \tag{3.3}$$

and thus a real null vector can be written as $\kappa^\alpha \bar{\kappa}^{\dot{\alpha}}$ for some spinor κ^α. [1]

After complexifying the Minkowski space, we can now introduce the twistor space. The twistor space $\mathbb{PT}$ of the complexified Minkowski space is defined as an open subset of the complex projective space $\mathbb{CP}^3$ [2]. This $\mathbb{CP}^3$ is described by homogeneous coordinates $Z^A = (Z^1, Z^2, Z^3, Z^4)$, in which the coordinates could never all vanish $Z^A \neq (0,0,0,0)$, and is invariant under projective rescaling $rZ^A \sim Z^A$ for all non-zero $r \in \mathbb{C}$.

The coordinate of the twistor space is chosen to be the two spinors with opposite chirality:

$$Z^A = (\mu^{\dot{\alpha}}, \lambda_\alpha) . \tag{3.4}$$

The next step is to define a relationship between the twistor space and the complexified Minkowski space. This is called the twistor correspondence, as it is not a "map" by its non-locality. The relation that link the coordinates of these two spaces together is

$$\mu^{\dot{\alpha}} = x^{\alpha\dot{\alpha}} \lambda_\alpha , \tag{3.5}$$

known as the incidence relation.

The incidence relation tell us that by fixing a point $x \in \mathcal{M}_{\mathbb{C}}$, the "matrix" $x^{\alpha\dot{\alpha}}$ is a coefficient in a linear equation relating $\mu^{\dot{\alpha}}$ and λ_α. Since $\mathbb{PT}$ is a projective space, this

[1] For other signature, see [16].

[2] The selection of the open subset of $\mathbb{CP}^3$ depends on the signature of spacetime.

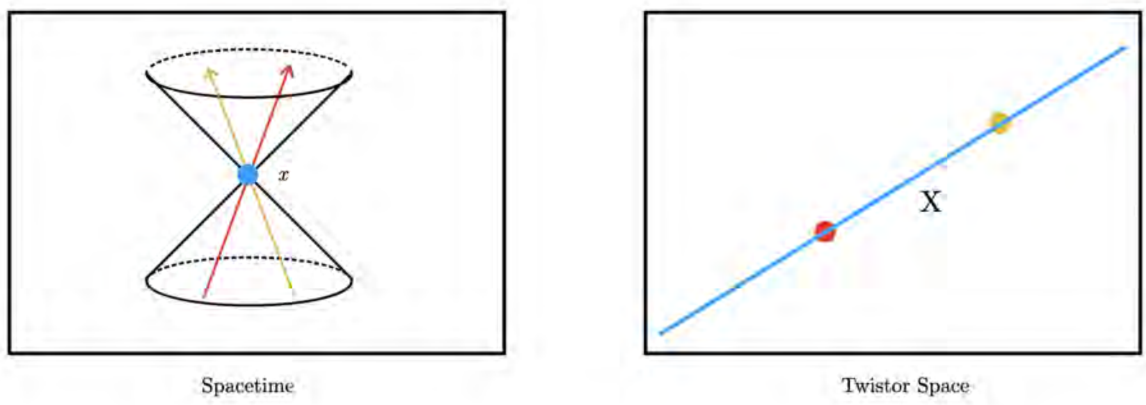


Figure 3.1: A picture showing the correspondence between a point $x \in \mathcal{M}_{\mathbb{C}}$ to a $\mathbb{CP}^1$ line $X \in \mathbb{PT}$, the two rays in Minkowski space is represented by two points on the twistor space, connected by the line X.

"plane" we have fixed is actually a $\mathbb{CP}^1$ line or a Riemann sphere [3]. So a point in the complexified Minkowski space corresponds to a linearly (and holomorphic, since there is no conjugation) embedded Riemann sphere in twistor space. Note that this relationship is non-local as a point is described by an extended object.

For the other way round, we can ask about what the point in twistor space is corresponding to. A point $Z \in \mathbb{PT}$ can be described by the intersection of two lines X and Y, described by the incidence relations

$$X \to \mu^{\dot{\alpha}} = x^{\alpha\dot{\alpha}}\lambda_\alpha \quad \text{and} \quad Y \to \mu^{\dot{\alpha}} = y^{\alpha\dot{\alpha}}\lambda_\alpha \,, \tag{3.6}$$

for $x, y \in \mathcal{M}_{\mathbb{C}}$. The intersection of these two equations yield

$$(x - y)^{\alpha\dot{\alpha}}\lambda_\alpha = 0 \,, \tag{3.7}$$

meaning that $(x-y)^{\alpha\dot{\alpha}} = 0$ or $(x-y)^{\alpha\dot{\alpha}}$ must be proportional to λ_α. So if x and y are two different points on $\mathcal{M}_{\mathbb{C}}$, the difference must obey

$$(x - y)^{\alpha\dot{\alpha}} = \tilde{\lambda}^{\dot{\alpha}}\lambda^\alpha \,, \tag{3.8}$$

where the spinor $\tilde{\lambda}^{\dot{\alpha}}$ is arbitrary. This mean that the separation is null-like. Therefore, the lines in twistor space will intersect if and only if their corresponding points in $\mathcal{M}_{\mathbb{C}}$ are null-separated. The arbitrary spinor $\tilde{\lambda}^{\dot{\alpha}}$ traces out a plane that every vector tangent to the plane is null. We can now see that the non-locality is reflected in both ways. The point in $\mathcal{M}_{\mathbb{C}}$ corresponds to a line in $\mathbb{PT}$, and an intersection of two lines in $\mathbb{PT}$ captures the lightcone structure in $\mathcal{M}_{\mathbb{C}}$ as the intersection indicates that the points are null-separated. Thus the conformal structures (lightcones) in $\mathcal{M}_{\mathbb{C}}$ is encoded by holomorphic structures (Riemann spheres) in $\mathbb{PT}$.

[3] A Riemann sphere can be obtained by a stereographic projection of a complex plane with a point at infinity $\mathbb{C}^2 \cup \{\infty\}$. The resulting object is a complex projective line $\mathbb{CP}^1$, where the additional point at infinity makes it compact and have the same topology with a sphere, so $\mathbb{CP}^1$ is also called the Riemann sphere. See [17] for more mathematical details.

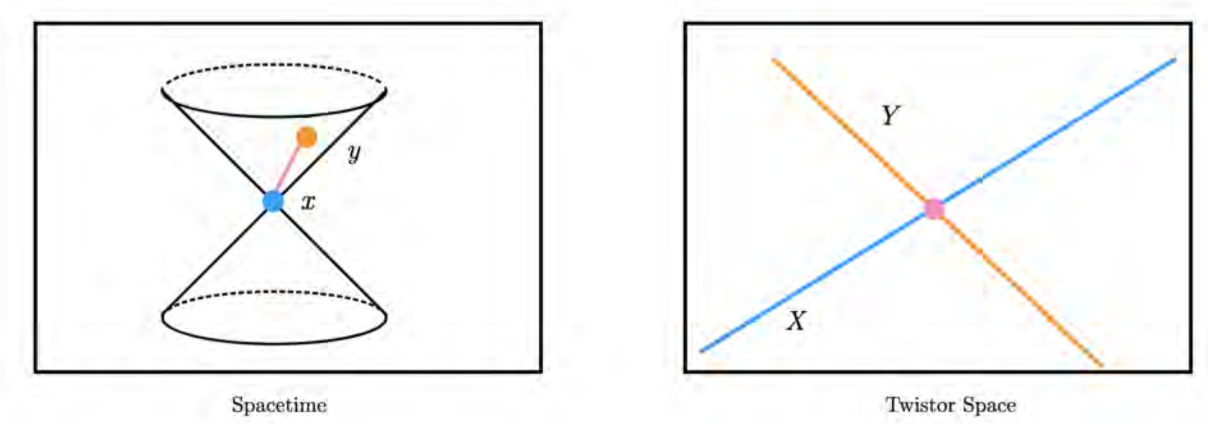


Figure 3.2: A picture showing the correspondence between an intersection of $\mathbb{CP}^1$ line X and Y on $\mathbb{PT}$ and the null-like points on the light cone in $\mathcal{M}_{\mathbb{C}}$

3.2 Penrose Transform

Now that we have the framework of the twistor theory to work with and understanding MHV and anti-MHV amplitudes requires some notion of fields, this section will discuss how the description the massless free fields on the twistor space is obtained.

For the spin one field, we can write the gauge field A_μ into an object with two spinor indices $A_{\alpha\dot{\alpha}}$. The field strength tensor in the spinor indices is then

$$F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \partial_{\alpha\dot{\alpha}}A_{\beta\dot{\beta}} - \partial_{\beta\dot{\beta}}A_{\alpha\dot{\alpha}}\,, \tag{3.9}$$

and is antisymmetric under $(\alpha\dot{\alpha}) \leftrightarrow (\beta\dot{\beta})$. If we want the antisymmetry under the exchange of separated spinor index (e.g. $\alpha \leftrightarrow \beta$ and $\dot{\alpha} \leftrightarrow \dot{\beta}$, we can achieve this antisymmetry by having the antisymmetry under the exchange of α and β and symmetric under $\dot{\alpha}$ and $\dot{\beta}$, or the other way round. This suggest that the field strength tensor can be decompose into two contributions

$$F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \frac{1}{2}\epsilon_{\alpha\beta}F^{\gamma}{}_{\dot{\alpha}\gamma\dot{\beta}} + \frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}F_{\alpha}{}^{\dot{\gamma}}{}_{\beta\dot{\gamma}} \tag{3.10}$$

one antisymmetric under $\alpha \leftrightarrow \beta$ and the other under $\dot{\alpha} \leftrightarrow \dot{\beta}$ due to the ϵ's. Meanwhile, the leftover of F are now symmetric under the exchange of the uncontracted indices, so we can define

$$\tilde{F}_{\dot{\alpha}\dot{\beta}} := \frac{1}{2}F^{\gamma}{}_{\dot{\alpha}\gamma\dot{\beta}}\,, \quad \text{and} \quad F_{\alpha\beta} := \frac{1}{2}F_{\alpha}{}^{\dot{\gamma}}{}_{\beta\dot{\gamma}}\,, \tag{3.11}$$

which are the self-dual (SD) and anti-self-dual (ASD) parts of the field strength tensor. The field strength tensor can then be written as

$$F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \epsilon_{\alpha\beta}\tilde{F}_{\dot{\alpha}\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}}F_{\alpha\beta}\,. \tag{3.12}$$

The 'self-duality' and 'anti-self-duality' come from the eigenvalue of each term under the duality operation, which is to contract $F_{\mu\nu}$ with a four-dimensional Levi-Cevita symbol to yield the dual field strength tensor. By contracting and changing the indices to spinor indices ($\mu, \nu, \rho, \sigma \to \alpha\dot{\alpha}, \beta\dot{\beta}, \gamma\dot{\gamma}, \delta\dot{\delta}$),

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu} = \frac{1}{2}\big(\epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\epsilon^{\dot{\alpha}\dot{\delta}}\epsilon^{\dot{\beta}\dot{\gamma}} - \epsilon^{\alpha\delta}\epsilon^{\beta\gamma}\epsilon^{\dot{\alpha}\dot{\gamma}}\epsilon^{\dot{\beta}\dot{\delta}}\big)(\epsilon_{\alpha\beta}\tilde{F}_{\dot{\alpha}\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}}F_{\alpha\beta}) = \epsilon^{\gamma\delta}\tilde{F}^{\dot{\gamma}\dot{\delta}} - \epsilon^{\dot{\gamma}\dot{\delta}}F^{\gamma\delta}\,, \tag{3.13}$$

in which the self-dual has the same sign and anti-self-dual becomes negative under this operation.

The Maxwell equations and the Bianchi identity for the decomposed field strength tensor are

$$\partial^{\dot{\alpha}}_{\beta}\tilde{F}_{\dot{\alpha}\dot{\beta}} + \partial^{\alpha}_{\dot{\beta}}F_{\alpha\beta} = 0 \quad \text{and} \quad \partial^{\dot{\alpha}}_{\beta}\tilde{F}_{\dot{\alpha}\dot{\beta}} - \partial^{\alpha}_{\dot{\beta}}F_{\alpha\beta} = 0\,. \tag{3.14}$$

Since the Bianchi identity must be satisfied for any field strength, this suggest that purely SD ($F_{\alpha\beta} = 0$) or purely ASD ($\tilde{F}_{\dot{\alpha}\dot{\beta}} = 0$) are consistent solutions to the equation of motion. The purely SD sector is identified with the positive helicity polarization and the purely ASD with the negative helicity polarization. A symmetric SD/ASD component will describe a Maxwell field if they satisfy

$$\partial^{\dot{\alpha}}_{\beta}\tilde{F}_{\dot{\alpha}\dot{\beta}} = 0 \quad \text{and} \quad \partial^{\alpha}_{\dot{\beta}}F_{\alpha\beta} = 0\,, \tag{3.15}$$

respectively. These are called the zero-rest-mass (z.r.m.) equations for the spin-1 fields. This procedure can be repeated to the field with any spin. In general, the z.r.m. equation of a field with helicity h is given by a linear partial differential equation with the field containing $2|h|$ spinor indices:

$$\begin{aligned}
&h > 0 \quad \tilde{\phi}_{\dot{\alpha}_1\ldots\dot{\alpha}_{2|h|}}\,, && \partial^{\beta\dot{\alpha}_1}\tilde{\phi}_{\dot{\alpha}_1\ldots\dot{\alpha}_{2|h|}} = 0\,,\\
&h = 0 \quad \Phi\,, && \Box\Phi = \partial^{\alpha\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\Phi = 0\,,\\
&h < 0 \quad \phi_{\alpha_1\ldots\alpha_{2|h|}}\,, && \partial^{\alpha_1\dot{\beta}}\phi_{\alpha_1\ldots\alpha_{2|h|}} = 0\,.
\end{aligned} \tag{3.16}$$

The Penrose transform is a way to transform the z.r.m. fields to an object on the twistor space. Let us first look at the fields with negative helicity. The correspondence of the twistor space and spacetime is non-local, so the first thing that we want to ensure under the correspondence of fields on twistor space is locality. We know that a point in $\mathcal{M}_{\mathbb{C}}$ corresponds to a $\mathbb{CP}^1$ line in $\mathbb{PT}$, and since the field is local, we need to integrate out the $\mathbb{CP}^1$ corresponding to the point on spacetime

$$\phi_{\alpha_1\ldots\alpha_{2s}}(x) = \int_{\mathbb{CP}^1}\big\langle\lambda d\lambda\big\rangle\ldots\,,$$

where $s = |h|$ and $\big\langle\lambda d\lambda\big\rangle$ is the natural measure on $\mathbb{CP}^1$. Next, we want the index structure of the field, so inside the integral we must have a product of $2s$-lambdas:

$$\phi_{\alpha_1\ldots\alpha_{2s}}(x) = \int_{\mathbb{CP}^1}\big\langle\lambda d\lambda\big\rangle\lambda_{\alpha_1}\ldots\lambda_{\alpha_{2s}}(\ldots)\,.$$

By having $2s$ of the λ variables and the measure, the $\mathbb{CP}^1$ weight of scaling $\lambda \to r\lambda$ is $2s + 2$. In order for the integral to be well-defined, the resulting object that we want must be independent of these $\mathbb{CP}^1$ scaling, so the integrand that we need must be some function of weight $-2s - 2$. Moreover, the measure is a complex $(1, 0)$-form. In order for this to be integrated over $\mathbb{CP}^1$, the integrand is required to be a $(1,1)$-form. We can now write the integral as

$$\phi_{\alpha_1...\alpha_{2s}}(x) = \int_{\mathbb{CP}^1} \langle \lambda d\lambda \rangle \lambda_{\alpha_1}...\lambda_{\alpha_{2s}} f(\lambda, \tilde{\lambda}) \,,$$

where the function $f(\lambda, \bar{\lambda})$ is of weight $-2s-2$ and is a $(0,1)$-form:

$$f(\lambda, \bar{\lambda}) = f^{\dot{\alpha}}(\lambda, \bar{\lambda}) d\bar{\lambda}_{\dot{\alpha}} \quad \text{and} \quad f(r\lambda, r\bar{\lambda}) = r^{-2s-2} f(\lambda, \bar{\lambda}) \tag{3.17}$$

and the $\bar{\lambda}$ is the conjugate of λ. The restriction of $f(\lambda, \bar{\lambda})$ on a $\mathbb{CP}^1$ line X is given by

$$f(Z, \bar{Z})\Big|_X = f(x^{\beta\dot{\alpha}}\lambda_\beta, \lambda_\alpha, \bar{x}^{\dot{\beta}\alpha}\bar{\lambda}_{\dot{\beta}}, \bar{\lambda}_{\dot{\alpha}}) \,. \tag{3.18}$$

Imposing the equation of motion yields

$$\partial^{\alpha_1\dot{\beta}} \phi_{\alpha_1...\alpha_{2|h|}} = \int_X \langle \lambda d\lambda \rangle \lambda_{\alpha_1}...\lambda_{\alpha_{2s}} \Big(\lambda^{\alpha_1} \frac{\partial f}{\partial \mu^{\dot{\beta}}} + \bar{\lambda}^{\dot{\beta}} \frac{\partial f}{\partial \bar{\mu}_{\alpha_1}} \Big) = 0 \,, \tag{3.19}$$

where the first term vanishes by $\lambda_{\alpha_1}\lambda^{\alpha_1} = 0$ and the second term will vanish if f is holomorphic (does not depend on the conjugate variables). Written in the language of complex geometry, f must satisfies $\bar{\partial} f = 0$.[4] And if $f = \bar{\partial} g$ for some function g, the field will directly vanish ($\phi = 0$) by using a similar argument. This mean that f is some $(0,1)$-form with weight $-2s-2$ obeying $\bar{\partial} f = 0$ and $f \neq \bar{\partial} g$. f is said to be in the cohomology group of weight $-2s-2$ on the twistor space.
The fields of other helicity are construct similarly, and they belongs to the cohomology group of weight $2h-2$ [16]. They are given by

$$\begin{aligned}
h > 0 \quad \tilde{\phi}_{\dot{\alpha}_1...\dot{\alpha}_{2h}} &= \int_{\mathbb{CP}^1} \langle \lambda d\lambda \rangle \frac{\partial}{\partial \mu^{\dot{\alpha}_1}} ... \frac{\partial}{\partial \mu^{\dot{\alpha}_{2h}}} f(\lambda, \tilde{\lambda})|_X \,, \\
h = 0 \quad \phi(x) &= \int_{\mathbb{CP}^1} \langle \lambda d\lambda \rangle f(\lambda, \tilde{\lambda})|_X \,, \\
h < 0 \quad \phi_{\alpha_1...\alpha_{2|h|}}(x) &= \int_{\mathbb{CP}^1} \langle \lambda d\lambda \rangle \lambda_{\alpha_1}...\lambda_{\alpha_{2s}} f(\lambda, \tilde{\lambda})|_X \,.
\end{aligned} \tag{3.20}$$

Now that we have the general construction for the Penrose transformation, we can look at the momentum eigenstates, which is the simplest case and it will be useful in the next chapters.

3.3 Momentum Eigenstates

A momentum eigenstate of a massless scalar field is given by

$$\phi(x) = e^{ik\cdot x} \,. \tag{3.21}$$

In the twistor theory, a momentum eigenstate of a field with helicity h can be obtained by the Penrose transform of the a twistor representative function $f^{[h]}(Z)$. The twistor representative function is given by

$$f^{[h]}(Z) = \int_{\mathbb{C}*} \frac{ds}{s^{2h-1}} \bar{\delta}^2(\kappa_\alpha - s\lambda_\alpha) \exp(is[\mu\tilde{\kappa}]) \,, \tag{3.22}$$

[4]The $\bar{\partial}$ is the Dolbeault operator, where its form on the twistor space is $\bar{\partial} = d\bar{Z}^{\bar{A}} \frac{\partial}{\partial \bar{Z}^{\bar{A}}}$.

where κ_α and $\tilde{\kappa}_{\dot\alpha}$ are constant spinors, and the delta function is defined by

$$\bar{\delta}^2(\lambda_\alpha) = \bigwedge_{\alpha=0,1} \bar{\partial}\Big(\frac{1}{\lambda_\alpha}\Big) . \tag{3.23}$$

To see that the twistor representative given above have the correct scaling of the twistor, we scale $Z \to cZ$. The scaling will appear in the delta function and the exponential, and the argument of these function must scale as 1, so we must treat the s with weight -1. Doing so, the integral is observed to have a weight of $2h-2$ from the measure. This representative can be checked that it belongs to the cohomology class of weight $2h-2$, by checking that it satisfies $\bar{\partial} f = 0$ and cannot be written as $\bar{\partial} g$ for some function g.
The first condition $\bar{\partial} f = 0$ can be directly seen by just applying the derivative :

$$\bar{\partial} f^{[h]}(Z) = d\bar{Z}^{\bar{A}} \frac{\partial f^{[h]}(Z)}{\partial \bar{Z}^{\bar{A}}} = \Big(d\bar{\lambda}^\alpha \frac{\partial}{\partial \bar{\lambda}^\alpha} + d\bar{\mu}^{\dot\alpha} \frac{\partial}{\partial \bar{\mu}_{\dot\alpha}}\Big) \int_{\mathbb{C}*} \frac{ds}{s^{2h-1}} \bar{\delta}^2(\kappa_\alpha - s\lambda_\alpha) e^{(is[\mu\tilde{\kappa}])} = 0 , \tag{3.24}$$

where the since the representative is independent of $\bar{\mu}$ and the wedge product of $d\bar{\lambda}$ into the delta function vanishes since the delta function also carry a $(0,2)$- form. The second condition can be seen by writing the Penrose transform of a derivative of a function, this will cause the field to become zero directly.
Before doing a Penrose transform, the delta function can be written in a more convenient way by choosing a normalize basis for the $(0,1)$ form as

$$e_0^\alpha = \kappa^\alpha \quad \text{and} \quad e_1^\alpha = \frac{a^\alpha}{\langle \kappa a \rangle} , \tag{3.25}$$

for some constant spinor a^α. With this choice of basis, the delta function can be rewritten as

$$\begin{aligned} \bar{\delta}^2(\kappa_\alpha - s\lambda_\alpha) &= \bigwedge_{\alpha=0,1} \bar{\partial}\Big(\frac{1}{\kappa_\alpha - s\lambda_\alpha}\Big) = \bar{\partial}\left(\frac{1}{-s\,\langle \kappa\lambda\rangle}\right) \bar{\partial}\left(\frac{1}{-1 + s\frac{\langle a\lambda\rangle}{\langle a\kappa\rangle}}\right) \\ &= \frac{1}{s} \bar{\delta}\big(\langle \kappa\lambda\rangle\big) \bar{\delta}\left(s - \frac{\langle a\kappa\rangle}{\langle a\lambda\rangle}\right) \frac{\langle a\kappa\rangle}{\langle a\lambda\rangle} . \end{aligned} \tag{3.26}$$

For a scalar field, the Penrose transform of the twistor representative is given by

$$\phi(x) = \int_{\mathbb{CP}^1} \langle \lambda d\lambda \rangle f^{[0]}(\lambda, \tilde{\lambda})|_X , \tag{3.27}$$

substituting the twistor representative gives

$$\phi(x) = \int_{\mathbb{CP}^1} \langle \lambda d\lambda \rangle \int_{\mathbb{C}*} sds\, \bar{\delta}^2(\kappa_\alpha - s\lambda_\alpha) \exp(is[\mu\tilde{\kappa}])|_X . \tag{3.28}$$

On the support of the delta functions, $s\lambda_\alpha = \kappa_\alpha$ and under the restriction on X, the twistor variable satisfies the incidence relation $\mu^{\dot\alpha} = x^{\alpha\dot\alpha}\lambda_\alpha$, and on the support this gives

$$\exp(ix^{\alpha\dot\alpha}\kappa_\alpha\tilde{\kappa}_{\dot\alpha}) = \exp(ik \cdot x) ,$$

for $k^{\alpha\dot{\alpha}} = p_\alpha \tilde{p}_{\dot{\alpha}}$. So now we have

$$
\begin{aligned}
&= \int_{\mathbb{CP}^1} \langle \lambda d\lambda \rangle \int_{\mathbb{C}*} sds\, \bar{\delta}^2(\kappa_\alpha - s\lambda_\alpha) \exp(ik\cdot x) \\
&= \int_{\mathbb{CP}^1} \langle \lambda d\lambda \rangle \int_{\mathbb{C}*} sds\, \frac{1}{s}\bar{\delta}\big(\langle \kappa\lambda \rangle\big) \bar{\delta}\left(s - \frac{\langle a\kappa \rangle}{\langle a\lambda \rangle}\right) \frac{\langle a\kappa \rangle}{\langle a\lambda \rangle} \exp(ik\cdot x) \\
&= \exp(ik\cdot x) \int_{\mathbb{CP}^1} \langle \lambda d\lambda \rangle \bar{\delta}\big(\langle \kappa\lambda \rangle\big) \frac{\langle a\kappa \rangle}{\langle a\lambda \rangle}\,.
\end{aligned}
\tag{3.29}
$$

Evaluating this integral can be done by choosing a basis $\lambda = (1, z)$, this simplifies the integral into

$$
\int_{\mathbb{CP}^1} dz \bar{\delta}\big(z\kappa_0 - \kappa_1\big) \frac{\langle a\kappa \rangle}{a_0 z - a_1} = \int_{\mathbb{CP}^1} dz \bar{\delta}\Big(z - \frac{\kappa_1}{\kappa_0}\Big) \frac{\langle a\kappa \rangle}{\kappa_0(a_0 z - a_1)} = 1 \tag{3.30}
$$

This gives the spin zero momentum eigenstate

$$
\phi(x) = e^{ik\cdot x}\,. \tag{3.31}
$$

For negative helicity, the Penrose transform of the twistor representative into z.r.m. field is given by

$$
\phi_{\alpha_1 \ldots \alpha_{2|h|}}(x) = \int_{\mathbb{CP}^1} \langle \lambda d\lambda \rangle \lambda_{\alpha_1} \ldots \lambda_{\alpha_{2s}} f^{[h]}(\lambda, \tilde{\lambda})|_X\,, \tag{3.32}
$$

substituting the twistor representative in gives

$$
\begin{aligned}
\phi_{\alpha_1 \ldots \alpha_{2|h|}}(x) &= \int_X \langle \lambda d\lambda \rangle \lambda_{\alpha_1} \ldots \lambda_{\alpha_{2s}} \int_{\mathbb{C}*} \frac{ds}{s^{2h-1}} \bar{\delta}^2(\kappa_\alpha - s\lambda_\alpha) \exp(is[\mu\tilde{\kappa}])|_X \\
&= \int_X \langle \lambda d\lambda \rangle\, sds\, \kappa_{\alpha_1} \ldots \kappa_{\alpha_{2s}} \bar{\delta}^2(\kappa_\alpha - s\lambda_\alpha) \exp(ik\cdot x) \\
&= \kappa_{\alpha_1} \ldots \kappa_{\alpha_{2s}} \exp(ik\cdot x) \int_X \langle \lambda d\lambda \rangle\, sds\, \bar{\delta}^2(\kappa_\alpha - s\lambda_\alpha)\,.
\end{aligned}
\tag{3.33}
$$

The integral is the same for the spin zero field, giving

$$
\phi_{\alpha_1 \ldots \alpha_{2|h|}} = \kappa_{\alpha_1} \ldots \kappa_{\alpha_{2s}} \exp(ik\cdot x) \tag{3.34}
$$

For positive helicity,

$$
\begin{aligned}
\tilde{\phi}_{\dot{\alpha}_1 \ldots \dot{\alpha}_{2h}} &= \int_{\mathbb{CP}^1} \langle \lambda d\lambda \rangle \frac{\partial}{\partial \mu^{\dot{\alpha}_1}} \cdots \frac{\partial}{\partial \mu^{\dot{\alpha}_{2h}}} f^{[h]}(\lambda, \tilde{\lambda})|_X \\
&= \tilde{\kappa}_{\dot{\alpha}_1} \ldots \tilde{\kappa}_{\dot{\alpha}_{2h}} \exp(ik\cdot x) \int_{\mathbb{CP}^1} \langle \lambda d\lambda \rangle\, sds\, \bar{\delta}^2(\kappa_\alpha - s\lambda_\alpha) \\
&= \tilde{\kappa}_{\dot{\alpha}_1} \ldots \tilde{\kappa}_{\dot{\alpha}_{2h}} \exp(ik\cdot x)\,.
\end{aligned}
\tag{3.35}
$$

This shows the construction of z.r.m. fields from twistor space for arbitrary helicity, which will be crucial for the next chapter.

Chapter 4

RSVW formula

The formula for general helicity amplitude is obtained using the twistor theory by [5] and [6]. In this chapter, a brief review of the RSVW formula will be presented, followed by a derivation of the refined scattering equation from the constraints in the RSVW.

Before directly jumping into the RSVW formula, let us look at the motivation Witten gave in [5] that allow us to look at the MHV amplitude on the twistor space. First, recall the n-points MHV superamplitude obtained using the Grassmannian variable in (C.14),

$$\mathcal{A}_n^{\text{MHV}} = \frac{\delta^8(\tilde{\mathcal{Q}})\delta^4(\sum_{i=1}^n \kappa_i^\alpha \tilde{\kappa}_i^{\dot{\alpha}})}{\langle 12\rangle \langle 23\rangle \ldots \langle n1\rangle}$$

where $\delta^8(\tilde{Q}) = \frac{1}{2^4}\prod_{A=1}^4 \sum_{i,j=1}^n \langle ij\rangle \eta_{iA}\eta_{jA}$. We can obtain the Parke-Taylor formula by applying the derivatives with respect to η_i and η_j on the superamplitude.

On twistor space, the transformation of the variables under the projective scaling are of the form

$$\lambda_\alpha \sim t\lambda_\alpha \quad \text{and} \quad \mu^{\dot{\alpha}} \sim t\mu^{\dot{\alpha}}\,, \tag{4.1}$$

but for the momentum, the little group scaling scale the momentum's spinor component, $p_i = \lambda_{i\alpha}\tilde{\lambda}_{i\dot{\alpha}}$, as

$$\lambda_\alpha \sim t\lambda_\alpha \quad \text{and} \quad \tilde{\lambda}_{\dot{\alpha}} \sim t^{-1}\tilde{\lambda}_{\dot{\alpha}}\,. \tag{4.2}$$

The right-handed is off by the inverse of the scaling, but under a twistor Fourier transform:

$$\tilde{\lambda}_{\dot{\alpha}} \to \frac{\partial}{\partial\mu^{\dot{\alpha}}}\,, \quad \text{and} \quad \frac{\partial}{\partial\tilde{\lambda}_{\dot{\alpha}}} \to \mu^{\dot{\alpha}}\,, \tag{4.3}$$

the scaling now match the scaling of the twistors. Performing a twistor Fourier transform on the amplitude, (C.14) becomes

$$\mathcal{A}_n^{\text{MHV}}\Big|_{\mathbb{PT}} = \int \prod_{i=1}^n \frac{d^2\tilde{\lambda}_i}{(2\pi)^2} e^{i\sum_{i=1}^n[\mu_i\tilde{\lambda}_i]}\delta^4\left(\sum_{i=1}^n \lambda_{i\alpha}\tilde{\lambda}_{i\dot{\alpha}}\right)\frac{\delta^8(\tilde{Q})}{\langle 12\rangle\ldots\langle n1\rangle}\,. \tag{4.4}$$

The delta functions $\delta^4(\sum_{i=1}^n \lambda_{i\alpha}\tilde{\lambda}_{i\dot{\alpha}})$ can be written as an integral form:

$$(2\pi)^4\delta^4(\sum_{i=1}^n \lambda_{i\alpha}\tilde{\lambda}_{i\dot{\alpha}}) = \int d^4x_{\alpha\dot{\alpha}} e^{-ix_{\alpha\dot{\alpha}}\sum_{i=1}^n \lambda_{i\alpha}\tilde{\lambda}_{i\dot{\alpha}}}\,, \tag{4.5}$$

which can be combined with the other exponential factor and result in a delta function

$$
\begin{aligned}
\mathcal{A}_n^{\text{MHV}}\Big|_{\mathbb{PT}} &= \int \prod_{i=1}^{n} \frac{d^2\tilde{\lambda}_i}{(2\pi)^2} e^{i\sum_{i=1}^{n}[\mu_i\tilde{\lambda}_i]} \frac{1}{(2\pi)^4} \int d^4x_{\alpha\dot{\alpha}} e^{-ix_{\alpha\dot{\alpha}}\sum_{i=1}^{n}\lambda_{i\alpha}\tilde{\lambda}_{i\dot{\alpha}}} \frac{\delta^8(\tilde{Q})}{\langle 12\rangle ... \langle n1\rangle} \\
&= \int \prod_{i=1}^{n} \frac{d^2\tilde{\lambda}_i}{(2\pi)^6} d^4x_{\alpha\dot{\alpha}} e^{i(\sum_{i=1}^{n}[\mu_i\tilde{\lambda}_i] - x_{\alpha\dot{\alpha}}\sum_{i=1}^{n}\lambda_{i\alpha}\tilde{\lambda}_{i\dot{\alpha}})} \frac{\delta^8(\tilde{Q})}{\langle 12\rangle ... \langle n1\rangle} \\
&= \frac{1}{(2\pi)^4} \int d^4x_{\alpha\dot{\alpha}} \prod_{i=1}^{n} \bar{\delta}^2(\mu_i^{\dot{\alpha}} - x^{\alpha\dot{\alpha}}\lambda_{i\alpha}) \frac{\delta^8(\tilde{Q})}{\langle 12\rangle ... \langle n1\rangle} .
\end{aligned} \tag{4.6}
$$

This tell us that the MHV amplitude is supported on a linear equation, that is the incidence relation for each particle. This degree one curve in twistor space is a feature that is realized to be important and extended further. The resulting amplitude formula, the RSVW formula, has shown that the N^{k-2} MHV at tree-level corresponds to a genus zero, degree $k-1$ curve on twistor space.

4.1 RSVW Formula

Now that we have seen that the MHV amplitude is an integral on the support of a degree 1 curve.The general expression of n particle scattering amplitudes at tree-level for gauge theory as given in [5] and [6] is

$$
\mathcal{A}_n = \int_{\mathcal{M}_{0,n}} \frac{\prod_{i=1}^{n} d\sigma_i}{\text{vol}(SL(2,\mathbb{C}) \times U(1))} \int_{Z(\sigma)} \prod_{r=0}^{d} d^{4|\mathcal{N}}\xi_r^I \frac{1}{\sigma_{12}...\sigma_{n1}} \prod_{i=1}^{n} a_i(Z(\sigma_i)) , \tag{4.7}
$$

where the integration over σ_i is taken over the moduli space $\mathcal{M}_{0,n}$ of the Riemann sphere with n marked points[1] [2], with the i^{th} marked point denoted σ_i, and the i_r^I are the parameters or "moduli" of the curve $Z(\sigma)$ of degree d. The σ_{ij} are just a short-hand notation for $\sigma_i - \sigma_j$, which will appear frequently in this work. The a_i's are the momentum eigenstates on twistor space given by (3.22)

$$
a_i(Z(\sigma_i)) = \int \frac{dt_i}{t_i} \bar{\delta}^2(\kappa_i - t_i\lambda(\sigma_i)) e^{i[\mu(\sigma_i)\tilde{\kappa}_i]t_i} , \tag{4.8}
$$

with the momentum $k_i^{\alpha\dot{\alpha}} = \kappa_i^\alpha \tilde{\kappa}_i^{\dot{\alpha}}$ and the twistor Z is now a function mapping the Riemann sphere to the twistor space $Z(\sigma) = (\mu^{\dot{\alpha}}(\sigma), \lambda_\alpha(\sigma), \chi^A(\sigma))$ [3]. The factor of $\text{vol}(SL(2,\mathbb{C}) \times U(1))$ denotes the quotient of the $SL(2,\mathbb{C})$ Mobius invariance of σ's on the Riemann sphere and $U(1)$ re-scaling invariance of the t's.

[1] the parameter space of the n-marked point sphere a.k.a the different ways we can move those marked points on the sphere

[2] Here, the marked Riemann sphere comes from the original derivation of the formula that uses a worldsheet model, where the worldsheet is compactified into a marked Riemann sphere. This worldsheet model is a maximally supersymmetric string theory in twistor space, called the twistor string theory. See [5] and [18].

[3] This is, in fact, the map from a worldsheet to twistor space as its target space in the twistor string theory.

The twistor curve that is integrated over is a degree d curve expressed by

$$Z^I(\sigma) = \sum_{r=0}^{d} \xi_r^I C_r(\sigma)\,, \tag{4.9}$$

where ξ_r^I are the parameters of the curve and $C_r(\sigma)$ are the degree r polynomials $\sim \sigma^r$ that forms a basis of the curve. Here, the curve that we want to express is the curve of twistor on the Riemann sphere, where the parameters in ξ_r^I are defined as

$$\xi_r^I = (\mu_r^{\dot\alpha}, \rho_{r\alpha}, \chi_r^A)\,, \tag{4.10}$$

and

$$\mu^{\dot\alpha}(\sigma) = \sum_{r=0}^{d} \mu_r^{\dot\alpha} C_r(\sigma)\,, \quad \lambda_\alpha(\sigma) = \sum_{r=0}^{d} \rho_{r\alpha} C_r(\sigma)\,, \quad \text{and} \quad \chi^A(\sigma) = \sum_{r=0}^{d} \chi_r^A C_r(\sigma)\,. \tag{4.11}$$

Using this expression for the "twistor" variables, we can write (4.7) as

$$\begin{aligned} \mathcal{A}_n = & \int_{\mathcal{M}_{0,n}} \frac{\prod_{i=1}^n d\sigma_i dt_i/t_i}{\text{vol}(SL(2,\mathbb{C})\times U(1))} \int_{Z(\sigma)} \prod_{r=0}^{d} d^2\rho_r d^2\mu_r d^{0|\mathcal{N}}\chi_r \frac{\text{Tr}(...)}{\sigma_{12}...\sigma_{n1}} \\ & \times \left[\prod_{j=1}^{n} \bar\delta^2\big(\kappa_j^\alpha - t_j \lambda^\alpha(\sigma_j)\big) \exp(i([\mu(\sigma_i)\tilde\kappa_i] + \chi(\sigma_i)^A \chi_A i) t_i) \right] \\ = & \int_{\mathcal{M}_{0,n}} \frac{\prod_{i=1}^n d\sigma_i dt_i/t_i}{\text{vol}(SL(2,\mathbb{C})\times U(1))} \int_{Z(\sigma)} \prod_{r=0}^{d} d^2\rho_r \frac{\text{Tr}(...)}{\sigma_{12}...\sigma_{n1}} \\ & \times \left[\prod_{j=1}^{n} \bar\delta^2\big(\kappa_j^\alpha - t_j \lambda^\alpha(\sigma_j)\big) \prod_{a=0}^{d} \bar\delta^{2|\mathcal{N}}\Big(\sum_{i=1}^{n} t_i \tilde\kappa_i^{\dot\alpha} C_a(\sigma_i) | \sum_{i=1}^{n} t_i \chi_A C_a(\sigma_i)\Big) \right]. \end{aligned} \tag{4.12}$$

As a sanity check, we can see that there are $n+n+2(d+1)-(3+1) = 2n+2(d+1)-4$ integration variable coming from σ's, t's, and ρ's, respectively, and the $-(3+1)$ are from the invariance of the variables. Meanwhile we have a total of $2n+2(d+1)$ delta function constraints (not counting the supersymmetric). So we are left with exactly 4 delta functions, which after evaluation they are the momentum conservation.

In the next part, the constraints of the RSVW formula will be simplified into nicer and more managable equations. For simplicity, the supersymmetric part will be omitted for now, since the supersymmetric extension is straightforward. It is true that the RSVW formula is derived for a supersymmetric theory, but since we are looking at the bosonic tree level amplitudes with gluons as external particles, it restricted fermions to not contribute.

4.2 Refined Scattering Equations

The RSVW formula can be simplified using the link variables as presented in [19]. The choice of $C_r(\sigma)$ were made to be

$$C_{i-1}(\sigma) = \prod_{\substack{l\neq i \\ l=1}}^{k} \frac{\sigma - \sigma_l}{\sigma_{il}}\,, \tag{4.13}$$

where k is the number of negative helicity particles and $k = d + 1$, such that

$$C_{i-1}(\sigma_j) = \begin{cases} \delta_{ij} & ; j = 1, ..., k\,, \\ \prod_{\substack{l \neq i \\ l=1}}^{k} \frac{\sigma_{jl}}{\sigma_{il}} & ; j = k+1, ..., n\,. \end{cases} \tag{4.14}$$

This allow us to simplify the constraints further. For convenience, consider only the constraints of $\lambda(\sigma)$ under this choice of $C_r(\sigma)$. The product from 1 to n can be split into $1, ..., k$ and $k+1, ..., n$:

$$\prod_{j=1}^{n} \bar{\delta}^2\big(\kappa_j^\alpha - t_j \lambda^\alpha(\sigma_j)\big) = \prod_{j=1}^{k} \bar{\delta}^2\big(\kappa_j^\alpha - t_j \rho_{j-1}^\alpha\big) \prod_{p=k+1}^{n} \bar{\delta}^2\big(\kappa_p^\alpha - t_p \lambda^\alpha(\sigma_p)\big)\,. \tag{4.15}$$

Integrating the delta functions with j indices constraints the variable $\rho_{j-1}^\alpha = \frac{\kappa_j^\alpha}{t_j}$ and this integration gives a Jacobian factor of $\mathcal{J} = \prod_{j=1}^{k} t_j^{-2}$, making the RSVW formula becomes

$$\begin{aligned} \mathcal{A}_n = \int_{\mathcal{M}_{0,n}} & \frac{\prod_{i=1}^{n} d\sigma_i dt_i/t_i}{\mathrm{vol}(SL(2,\mathbb{C}) \times U(1))} \frac{\mathrm{Tr}(...)}{\sigma_{12}...\sigma_{n1}} \mathcal{J} \prod_{p=k+1}^{n} \bar{\delta}^2\Big(\kappa_p^\alpha - t_p \sum_{i=1}^{k} \frac{\kappa_i^\alpha}{t_i} \frac{\prod_{l\neq i}^{k} \sigma_{pl}}{\prod_{l\neq i}^{k} \sigma_{il}}\Big) \\ & \times \prod_{i=1}^{k} \bar{\delta}^{2|\mathcal{N}}\Big(t_i \tilde{\kappa}_i^{\dot\alpha} + \sum_{p=k+1}^{n} t_p \tilde{\kappa}_p^{\dot\alpha} \frac{\prod_{l\neq i}^{k} \sigma_{pl}}{\prod_{l\neq i}^{k} \sigma_{il}}\Big)\,. \end{aligned} \tag{4.16}$$

With a change of variable

$$\text{for} \quad i = 1, ..., k : u_i = \frac{1}{t_i \prod_{l\neq i}^{k} \sigma_{il}} \quad \text{and for} \quad p = k+1, ..., n : u_p = \prod_{l=1}^{k} \sigma_{pl} t_p\,, \tag{4.17}$$

the constraints become

$$\begin{aligned} & \prod_{p=k+1}^{n} \bar{\delta}^2\Big(\kappa_p^\alpha - t_p \sum_{i=1}^{k} \frac{\kappa_i^\alpha}{t_i} \frac{\prod_{l\neq i}^{k} \sigma_{pl}}{\prod_{l\neq i}^{k} \sigma_{il}}\Big) \prod_{i=1}^{k} \bar{\delta}^{2|\mathcal{N}}\Big(t_i \tilde{\kappa}_i^{\dot\alpha} + \sum_{p=k+1}^{n} t_p \tilde{\kappa}_p^{\dot\alpha} \frac{\prod_{l\neq i}^{k} \sigma_{pl}}{\prod_{l\neq i}^{k} \sigma_{il}}\Big) \\ & = \prod_{p=k+1}^{n} \bar{\delta}^2\Big(\kappa_p^\alpha - t_p \sum_{i=1}^{k} \frac{\kappa_i^\alpha}{t_i} \frac{\prod_{l=1}^{k} \sigma_{pl}}{\sigma_{pi} \prod_{l\neq i}^{k} \sigma_{il}}\Big) \prod_{i=1}^{k} t_i^{-2} \bar{\delta}^{2|\mathcal{N}}\Big(\tilde{\kappa}_i^{\dot\alpha} + \sum_{p=k+1}^{n} t_p \frac{\tilde{\kappa}_p^{\dot\alpha}}{t_i} \frac{\prod_{l=1}^{k} \sigma_{pl}}{\sigma_{pi} \prod_{l\neq i}^{k} \sigma_{il}}\Big) \\ & = \mathcal{J} \prod_{p=k+1}^{n} \bar{\delta}^2\Big(\kappa_p^\alpha - u_p \sum_{i=1}^{k} \frac{u_i \kappa_i^\alpha}{\sigma_{pi}}\Big) \prod_{i=1}^{k} \bar{\delta}^{2|\mathcal{N}}\Big(\tilde{\kappa}_i^{\dot\alpha} + u_i \sum_{p=k+1}^{n} \frac{u_p \tilde{\kappa}_p^{\dot\alpha}}{\sigma_{pi}}\Big)\,. \end{aligned} \tag{4.18}$$

Thus, the RSVW formula becomes

$$A_n = \int_{\mathcal{M}_{0,n}} \frac{\prod_{i=1}^{n} d\sigma_i du_i/u_i}{\mathrm{vol}(SL(2,\mathbb{C}) \times U(1))} \prod_{i=1}^{k} \bar{\delta}^2\big(\tilde{\kappa}_i^{\dot\alpha} - u_i \tilde{\lambda}_i^{\dot\alpha}(\sigma_i)\big) \prod_{p=k+1}^{n} \bar{\delta}^{2|\mathcal{N}}\big(\kappa_p^\alpha - u_p \lambda(\sigma_p)\big) \mathcal{J}^2 \Big(\frac{\mathrm{Tr}(...)}{\sigma_{12}...\sigma_{n1}}\Big)\,, \tag{4.19}$$

where

$$\tilde{\lambda}^{\dot\alpha}(\sigma) = \sum_{p=k+1}^{n} \frac{u_p \tilde{\kappa}_p^{\dot\alpha}}{\sigma - \sigma_p} \quad \text{and} \quad \lambda^\alpha(\sigma) = \sum_{i=1}^{k} \frac{u_i \kappa_i^\alpha}{\sigma - \sigma_i}\,. \tag{4.20}$$

The Jacobian factor $\mathcal{J}^2$ cancels if we are in an $\mathcal{N} = 4$ SYM theory, due to the fact that the supersymmetric delta function of the supersymmetric part gives a factor of $\prod_{i=1}^{k} t_i^4$

when pulling the t_i's out. We can write this in a more compact form by defining the measure and the constraints as the 'reduced measure'

$$d\mu_{n,k} := \frac{\prod_{i=1}^{n} d\sigma_i du_i/u_i}{\text{vol}(SL(2,\mathcal{C}) \times U(1))} \prod_{i=1}^{k} \bar{\delta}^2\left(\tilde{\kappa}_i^{\dot{\alpha}} - u_i \tilde{\lambda}(\sigma_i)\right) \prod_{p=k+1}^{n} \bar{\delta}^{2|4}\left(\kappa_p^{\alpha} - u_p \lambda(\sigma_p) \middle| \chi_p - u_p \chi(\sigma_p)\right), \tag{4.21}$$

and the RSVW is now just

$$\mathcal{A}_n = \int d\mu_{n,k} \frac{\text{Tr}(...)}{\sigma_{12}...\sigma_{n1}}. \tag{4.22}$$

Note that it does not matter which specific particle is negative helicity, since the formula is invariant under which k particle would be negative helicity. Therefore the formula only depends on the number be of particles in each sector. This observation will be useful and explored in details for the next chapter.

The constraints of RSVW formula will be the point of focus in the next section. They are called the refined scattering equations as they are "refined" according to their MHV degree

$$\tilde{\mathcal{E}}_i^{\dot{\alpha}} = \tilde{\kappa}_i^{\dot{\alpha}} - u_i \tilde{\lambda}^{\dot{\alpha}}(\sigma_i) \quad \text{and} \quad \mathcal{E}_p^{\alpha} = \kappa_p^{\alpha} - u_p \lambda^{\alpha}(\sigma_p). \tag{4.23}$$

The name "scattering equations" came from another formalism by Cachazo, He, and Yuan (the CHY) where the amplitude is an integral over some constraints, and those constraints are called the scattering equations. This formalism will be discussed and made contact with in Chapter 6.

The refined scattering equation can be dotted with $\tilde{\kappa}_i$ and κ_p. Doing so gives the relation

$$\left[\tilde{\kappa}_i \tilde{\lambda}(\sigma_i)\right] = 0 \quad \text{and} \quad \left\langle \kappa_p \lambda(\sigma_p) \right\rangle = 0, \tag{4.24}$$

implying that λ and $\tilde{\lambda}$ defined on the Riemann sphere are parallel to the external spinors at the marked points.

On the support of the refined scattering equations, the momentum conservation is also satisfied:

$$\begin{aligned}
\sum_{i=1}^{n} k_i^{\alpha\dot{\alpha}} &= \sum_{i=1}^{k} \kappa_i^{\alpha} \tilde{\kappa}_i^{\dot{\alpha}} + \sum_{p=k+1}^{n} \kappa_p^{\alpha} \tilde{\kappa}_p^{\dot{\alpha}} \\
&= \sum_{i=1}^{k} \kappa_i^{\alpha} u_i \tilde{\lambda}_i^{\dot{\alpha}}(\sigma_i) + \sum_{p=k+1}^{n} u_p \lambda_p^{\alpha}(\sigma_p) \tilde{\kappa}_p^{\dot{\alpha}} \\
&= \sum_{i,p} \frac{u_i u_p}{\sigma_{ip}} \kappa_i^{\alpha} \tilde{\kappa}_p^{\dot{\alpha}} (1-1) = 0.
\end{aligned} \tag{4.25}$$

which is the momentum conservation in the four delta functions that is left when we count the number of delta functions.

The evaluation of this formula gives the sum of all solution of the constraints (refined scattering equation) that gives n-point amplitude with some Jacobian factors J:

$$\mathcal{A}_n = \sum_{\text{solutions}} \frac{1}{J} \frac{\text{Tr}(...)}{\sigma_{12}...\sigma_{n1}}. \tag{4.26}$$

Another point that is worth a remark is that the general form of the formula can be extend to other theory by changing the integrand, for example, in $4d$, the other possible theory is maximal supergravity:

$$\mathcal{M}_n = \int_{\mathcal{M}_{0,n}} d\mu_{n,k} \, \det{}' \mathcal{H} \, , \tag{4.27}$$

where the reduced measure is now defined with $\mathcal{N} = 8$ and $\mathcal{H}$ is a block diagonal matrix

$$\mathcal{H} = \begin{pmatrix} \mathbb{H} & 0 \\ 0 & \tilde{\mathbb{H}} \end{pmatrix} , \tag{4.28}$$

with

$$\mathbb{H}_{ij} = \frac{\left\langle ij \right\rangle}{\sigma_{ij}} \, , \qquad \mathbb{H}_{ii} = - \sum_{j=1, j\neq i}^{k} \mathbb{H}_{ij} \, , \tag{4.29}$$

$$\tilde{\mathbb{H}}_{pq} = \frac{\left[pq \right]}{\sigma_{pq}} \, , \qquad \tilde{\mathbb{H}}_{pp} = - \sum_{j=1, j\neq i}^{k} \tilde{\mathbb{H}}_{pq} \, , \tag{4.30}$$

for $i, j \in 1, ..., k$ and $p, q \in k+1, ..., n$ and $\det'$ is the reduced determinant where one takes the determinant of $\mathcal{H}$ with a row and a column removed from each of $\mathbb{H}$ and $\tilde{\mathbb{H}}$ [20].

Chapter 5

Number of Solutions at tree level

The number of solutions for the refined scattering equations in the RSVW formula of SYM was conjectured to be the Eulerian number $E(n-3, k-2)$ in [8] and the procedure to proof it was suggested in [21]. This was initially observed through directly counting the number of the solutions of each n and k. This chapter will provide some background on what the Eulerian numbers are, find the recursion relation for the refined scattering equations, prove that the number of solutions at tree level are the Eulerian numbers.

5.1 Eulerian Numbers

The Eulerian numbers $E(m,l)$ are defined as the number of permutations of n numbers with k descents, with $0 \leq l \leq m$ [22]. As an illustration, given a sequence of 3 numbers, 1,2,3, there are 4 permutations with 1 descents: 132, 312, 213, and 231. For 4 numbers, there are 11 permutations with 1 descents: 1243, 1324, 1342, 1423, 2134, 2314, 2341, 2413, 3124, 3412, and 4123. The Eulerian numbers for $n = 1-6$ is given by the following table.

m \l	0	1	2	3	4	5
1	1					
2	1	1				
3	1	4	1			
4	1	11	11	1		
5	1	26	66	26	1	
6	1	57	302	302	52	1

Table 5.1: The Eulerian numbers $E(m,l)$ for $m = 1, ..., 6$, adapted from [22].

The Eulerian numbers satisfy a number of properties:

1. $E(m,0) = E(m,m-1) = 1$,

2. $E(m,m) = 0$,

3. It is symmetric under changing m to $m-l$: $E(m,l) = E(m,m-l)$,

4. $\sum_{l=0}^{n} E(m,l) = m!$, and

5. It satisfies the recursion relation

$$E(m,l) = (m-l)E(m-1,l-1) + (l+1)E(m-1,l)\,. \tag{5.1}$$

The number of solutions to the refined scattering equations are observed to satisfies properties 1 to 4, with $m = n-3$ and $l = k-2$. The RSVW formula can be evaluate for the MHV and anti MHV sector, giving the MHV amplitude as in (C.14). The number of solutions for RSVW formula satisfy the first property as there are only one solution for MHV and anti-MHV. The second property can be seen directly from the fact that all plus/minus amplitude vanishes. The third property comes from the symmetry in switching negative and positive helicity particles. It is also known in [21] that for n particle amplitude, there are a total of $(n-3)!$ solutions, thus satisfy the fourth property. In this chapter, we will prove that the number of solutions for the refined scattering equations satisfy the same recursion relation for the Eulerian numbers.

5.2 Eulerian Numbers and the Number of Solutions for the Refined Scattering Equations

This section presents the proof that the number of solutions for the refined scattering equations are the Eulerian numbers. This will be shown by taking the soft limit of the n^{th} particle – taking the momentum $p_n \to 0$ – in the refined scattering equation. Then, the expression for σ_n and u_n is solved asymptotically using the method of dominance balance.[1] Counting the number of solution for each dominance balance gives a recursion relation for the number of solution, which is expected to be the Eulerian number's recursion relation.

Theorem 1. The number of solutions for the constraint equations from the RSVW formula,

$$\tilde{\mathcal{E}}_i^{\dot{\alpha}} = \sum_{p=k+1}^{n} \frac{u_i u_p}{\sigma_{ip}} \tilde{\kappa}_p^{\dot{\alpha}} - \tilde{\kappa}_i^{\dot{\alpha}} = 0 \quad \text{and} \quad \mathcal{E}_p^{\alpha} = \sum_{i=1}^{k} \frac{u_p u_i}{\sigma_{pi}} \kappa_i^{\alpha} - \kappa_p^{\alpha}\,, \tag{5.2}$$

for an n particles process with k negative helicity particles, are the Eulerian numbers $E(n-3, k-2)$.

The general strategy in proving the theorem is to construct a recursion relation for the constraints equation by looking at the soft limit of the n^{th} particle, explicitly this is

[1] See Appendix A for more detail of the method

given by taking $\kappa_n \to \epsilon\kappa_n$ and $\tilde{\kappa}_n \to \epsilon\tilde{\kappa}_n$, and $\epsilon \to 0$. Thus the equation (5.2) can be written as

$$\begin{aligned}
\forall i = 1, ..., k: \quad \tilde{\mathcal{E}}_i &= \sum_{p=k+1}^{n-1} \frac{u_i u_p}{\sigma_{ip}} \tilde{\kappa}_p + \epsilon \frac{u_i u_n}{\sigma_{in}} \tilde{\kappa}_n - \tilde{\kappa}_i \,, && (5.3)\\
\forall p = k+1, ..., n-1: \quad \mathcal{E}_p &= \sum_{i=1}^{k} \frac{u_p u_i}{\sigma_{pi}} \kappa_i - \kappa_p \,, && (5.4)\\
\mathcal{E}_n &= \sum_{i=1}^{k} \frac{u_n u_i}{\sigma_{ni}} \kappa_i - \epsilon\kappa_n \,, && (5.5)
\end{aligned}$$

under the rescaling of κ_n and $\tilde{\kappa}_n$, and $\mathcal{E}_n$ were written separately for convenience. Under the limit $\epsilon \to 0$, the refined scattering equations for $i \in 1, .., n-1$ will drop the term with $\tilde{\kappa}_n$, thus behave like the refined scattering equations for $n-1$ particles. The n^{th} particle's equation can then be solved for σ_n and u_n in each dominance balances.

Proof. In order to see the dominance balances explicitly, we can label the constraint equations as follows:

$$\tilde{\mathcal{E}}_i = \underbrace{\sum_{p=k+1}^{n-1} \frac{u_i u_p}{\sigma_{ip}} \tilde{\kappa}_p}_{\text{A}} + \underbrace{\epsilon \frac{u_i u_n}{\sigma_{in}} \tilde{\kappa}_n}_{\text{B}} - \underbrace{\tilde{\kappa}_i}_{\text{C}} \,. \tag{5.6}$$

We can see that for each choice of dominance balance, only two of them gives a solution for u_n and σ_n:

1. $A \sim 1, B \sim \epsilon$, and $C \sim 1$
2. $A \sim 1, B \sim 1$, and $C \sim 1$
3. $A \sim \epsilon^{-1}, B \sim \epsilon^{-1}$, and $C \sim 1$:
   This can be achieved by having $u_n \sim \epsilon^{-1}$ and $u_i \sim \epsilon^{-1}$. At the order of ϵ^{-1}, we have

   $$\sum_{p=k+1}^{n-1} \frac{u_i u_p}{\sigma_{ip}} \tilde{\kappa}_p^{\dot{\alpha}} + \frac{u_i u_n}{\sigma_{in}} \tilde{\kappa}_n^{\dot{\alpha}} = 0 \,,$$

   for $\dot{\alpha} = \dot{0}, \dot{1}$. As homogeneous equations of variable u_n and σ_{in}, we can see that its only solution is $u_n = 0$ and $\sigma_{in} = 0$, so $\sigma_i = \sigma_n$, therefore not a solution.
4. $A \sim \epsilon, B \sim 1$, and $C \sim 1$:
   This can be achieved having $u_n \sim \epsilon^{-2}$ and $u_i \sim \epsilon$. At the leading order, we can see that this gives a relation implying that $\tilde{\kappa}_n^{\dot{\alpha}}$ is proportional to $\tilde{\kappa}_i^{\dot{\alpha}}$. Since this must holds for all i, this imply that $k_i \cdot k_n = 0$. Therefore this dominance balance does not give a solution.

Now we have to prove that the two dominance balances give the same recursion relation for the number of solutions to the Eulerian numbers.

Balance 1 This choice of dominance balance can be achieved by $u_n = \epsilon \tilde{u}_n$. The refined scattering equations for $i \neq n$ under the limit $\epsilon \to 0$ look like the refined scattering equation for $n-1$ particle

$$\forall i = 1, ..., k: \quad \tilde{\mathcal{E}}_i = \sum_{p=k+1}^{n-1} \frac{u_i u_p}{\sigma_{ip}} \tilde{\kappa}_p + \epsilon \frac{u_i u_n}{\sigma_{in}} \tilde{\kappa}_n - \tilde{\kappa}_i \to \sum_{p=k+1}^{n-1} \frac{u_i u_p}{\sigma_{ip}} \tilde{\kappa}_p - \tilde{\kappa}_i \tag{5.7}$$

$$\forall p = k+1, ..., n-1: \quad \mathcal{E}_p = \sum_{i=1}^{k} \frac{u_p u_i}{\sigma_{pi}} \kappa_i - \kappa_p \,, \tag{5.8}$$

and they are independent of n, thus determine all the variables u_i, u_p, σ_i and σ_p. We can use the nth particle equation,

$$\mathcal{E}_n^\alpha = \epsilon \left(\tilde{u}_n \sum_{i=1}^{k} \frac{u_i}{\sigma_{ni}} \kappa_i^\alpha - \kappa_n^\alpha \right) \,, \tag{5.9}$$

to solve for the two variables σ_n and u_n as a function of the known variables, i.e. those u_i and σ_i of the $n-1$ particles. Notice that (5.9) is actually two equations in two variables, i.e. the equation $\mathcal{E}_n^0$ and $\mathcal{E}_n^1$, so we can separate them into two equations. We can solve for σ_n by dotting in $\sum_{j=1}^k u_j \sigma_{nj} \kappa_j$:

$$\begin{aligned} 0 &= \mathcal{E}_n^0 \Big(\sum_{j=1}^{k} u_j \sigma_{nj} \kappa_j^1 \Big) - \mathcal{E}_n^1 \Big(\sum_{j=1}^{k} u_j \sigma_{nj} \kappa_j^0 \Big) \\ &= \tilde{u}_n \left(\sum_{i,j}^{k} \frac{u_i u_j \kappa_i^0 \kappa_j^0}{\sigma_{ni} \sigma_{nj}} (1-1) \right) - \kappa_n^0 \sum_{j=1}^{k} u_i \sigma_{ni} \kappa_i^1 + \kappa_n^1 \sum_{j=1}^{k} u_i \sigma_{ni} \kappa_i^0 \\ &= \sum_{i}^{k} \frac{u_i \big\langle \kappa_n \kappa_i \big\rangle}{\sigma_{in}} \,. \end{aligned} \tag{5.10}$$

This can be rewritten to see that it is a $(k-1)$ degree polynomial in σ_n:

$$0 = \sum_{i}^{k} u_i \big\langle \kappa_n \kappa_i \big\rangle \prod_{j \neq i} \sigma_{jn} \,. \tag{5.11}$$

For the other equation, we can simply dotting in κ_k^α to (5.9), giving

$$\tilde{u}_n = \frac{\langle kn \rangle}{\sum_{i=1}^{k-1} \frac{u_i}{\sigma_{in}} \langle ki \rangle} \,. \tag{5.12}$$

Since all the variable on the right hand side is known, they uniquely determine a $\tilde{u}_n$ for a σ_n.

Here, the behaviour of the constraint equations of n particle under the soft limits will reduce to $n-1$ particle with k negative helicity particles, and there are $k-1$ solution for σ_n. Therefore in the recursion relation for the number of solutions, we pick up the term

$$(k-1)\mathcal{N}(n-1, k) \,. \tag{5.13}$$

Balance 2 The second choice of dominance balance is not so obvious to see. We can use the result of the following lemma to rewrite (5.5) in a more convenience way.

Lemma 1. *The measure $d\mu_{n,k}$ is invariant of the choice of negative helicity particle and depend on just the number of negative helicity particle.*

Proof. See Appendix B. □

Using lemma 1, the particle label k and n can be switched using ($B.4$) and ($B.7$) as

$$\upsilon_p = u_p \frac{\sigma_{pn}}{\sigma_{pk}} \qquad \upsilon_n = \frac{\sigma_{nk}}{u_n}\,,$$
$$\upsilon_i = u_i \frac{\sigma_{ik}}{\sigma_{in}} \qquad \upsilon_k = \frac{\sigma_{nk}}{u_k}\,,$$

so that particle n has negative helicity and k has positive helicity. The refined scattering equations under the switch are

$$\tilde{\mathcal{E}}_i^{\dot\alpha} = \sum_{p=k}^{n-1} \frac{\upsilon_i \upsilon_p}{\sigma_{ip}} \tilde\kappa_p^{\dot\alpha} - \tilde\kappa_i^{\dot\alpha}\,, \tag{5.14}$$

$\forall i = n, 1, ..., k-1$ and

$$\mathcal{E}_p^{\alpha} = \sum_{i=n}^{k-1} \frac{\upsilon_p \upsilon_i}{\sigma_{pi}} \kappa_i^{\alpha} - \kappa_p^{\alpha}\,, \tag{5.15}$$

$\forall p = k, ..., n-1$. The index of the summation is written from n to $k-1$ since the index is cyclic, and the variable u_i under the switch are denoted as υ_i.
Now, taking the soft limit of κ_n and $\tilde\kappa_n$, the equations become

$$\tilde{\mathcal{E}}_n^{\dot\alpha} = \sum_{p=k+1}^{n-1} \frac{\upsilon_n \upsilon_p}{\sigma_{np}} \tilde\kappa_p^{\dot\alpha} + \frac{\upsilon_n \upsilon_k}{\sigma_{nk}} \tilde\kappa_k^{\dot\alpha} - \epsilon\tilde\kappa_n^{\dot\alpha}\,, \tag{5.16}$$

$$\forall i = 1, ..., k-1: \quad \tilde{\mathcal{E}}_i^{\dot\alpha} = \sum_{p=k+1}^{n-1} \frac{\upsilon_i \upsilon_p}{\sigma_{ip}} \tilde\kappa_p^{\dot\alpha} + \frac{\upsilon_i \upsilon_k}{\sigma_{ik}} \tilde\kappa_k^{\dot\alpha} - \tilde\kappa_i^{\dot\alpha}\,, \tag{5.17}$$

$$\mathcal{E}_k^{\alpha} = \sum_{i=1}^{k-1} \frac{\upsilon_k \upsilon_i}{\sigma_{ki}} \kappa_i^{\alpha} + \epsilon \frac{\upsilon_k \upsilon_n}{\sigma_{kn}} \kappa_n^{\alpha} - \kappa_p^{\alpha}\,, \tag{5.18}$$

$$\forall p = k+1, ..., n-1: \quad \mathcal{E}_p^{\alpha} = \sum_{i=1}^{k-1} \frac{\upsilon_p \upsilon_i}{\sigma_{pi}} \kappa_i^{\alpha} + \epsilon \frac{\upsilon_p \upsilon_n}{\sigma_{pn}} \kappa_n^{\alpha} - \kappa_p^{\alpha}\,. \tag{5.19}$$

We can see that under the limit $\epsilon \to 0$, (5.18) and (5.19) will just drop the term with n^{th} particle out, and behave like constraint equations for $n-1$ particle. We can now use (5.16) to determine the number of solution of σ_n like in the proof above. The dominance balance can be acheived by having $\upsilon_n = \epsilon\tilde\upsilon_n$, making (5.16) become

$$\tilde{\mathcal{E}}_n^{\dot\alpha} = \epsilon \left(\sum_{p=k+1}^{n-1} \frac{\tilde\upsilon_n \upsilon_p}{\sigma_{np}} \tilde\kappa_p^{\dot\alpha} + \frac{\tilde\upsilon_n \upsilon_k}{\sigma_{nk}} \tilde\kappa_k^{\dot\alpha} - \tilde\kappa_n^{\dot\alpha} \right)\,. \tag{5.20}$$

The two equations in (5.20) can be separated to solve for $\tilde{\upsilon}_n$ and σ_n using the same trick. The equation for σ_n can be separate by dotting in $\tilde{\kappa}_n^{\dot{\alpha}}$ to (5.20), giving the relation

$$0 = \sum_{p=k}^{n-1} \frac{\tilde{\upsilon}_n \upsilon_p}{\sigma_{np}} \left[\tilde{\kappa}_p \tilde{\kappa}_n\right] , \tag{5.21}$$

that can be rewritten as an $n-k-1$ degree polynomial in σ_n

$$0 = \sum_{p=k}^{n-1} \tilde{\upsilon}_n \upsilon_p \left[\tilde{\kappa}_p \tilde{\kappa}_n\right] \prod_{q \neq p} \sigma_{nq} \, . \tag{5.22}$$

For the expression of $\tilde{\upsilon}_n$, we can dot in $\tilde{\kappa}_k^{\dot{\alpha}}$ and yield

$$\tilde{\upsilon}_n = \frac{[kn]}{\sum_{p=k+1}^{n-1} \frac{\upsilon_p}{\sigma_{np}} [kp]} \, . \tag{5.23}$$

It can be seen that υ_n can be uniquely determine for each σ_n. The behaviour of the constraint equations of n particle under the soft limits will reduce to $n-1$ external particles with $k-1$ negative helicity particles, with $n-k-1$ solution for σ_n. Therefore in the recursion relation for the number of solution, we pick up the term

$$(n-k-1)\mathcal{N}(n-1, k-1) \, . \tag{5.24}$$

Under the mapping (B.7) and (B.4), the choice of having $\upsilon_n = \epsilon \tilde{\upsilon}$ corresponds to $u_n \sim \epsilon^{-1}$. In the original configuration, this corresponds to having all the terms in (5.3) behave like order ~ 1, which is the second dominance balance.

The recursion relation for number of the solutions for the refined scattering equations with n external particles with k particles of negative helicity is

$$\mathcal{N}(n, k) = (n-k-1)\mathcal{N}(n-1, k-1) + (k-1)\mathcal{N}(n-1, k) \, . \tag{5.25}$$

This recursion relation starts at $\mathcal{N}(4, 2) = 1$ which is the same initial value as the Eulerian number $E(1, 0)$, identifying $m = n-3, l = k-2$. Since it starts on the same value and has the same recursion relation, the number of solutions to the refined scattering equations are the Eulerian numbers. □

From the result above, the reduced measure of n particle with k negative helicity particle can be written as

$$\begin{aligned} d\mu_{n,k} =& d\mu'_{n-1,k-1} \left(d\sigma_n \frac{d\upsilon_n}{\upsilon_n} \bar{\delta}\Big(\sum_{p=k}^{n-1} \frac{\upsilon_n \upsilon_p}{\sigma_{np}} [kp] - [kn] \Big) \bar{\delta}\Big(\sum_{p=k}^{n-1} \upsilon_n \upsilon_p [pn] \prod_{q \neq p}^{n-1} \sigma_{nq} \Big) J_{k-1} \right) \\ &+ d\mu_{n-1,k} \left(d\sigma_n \frac{du_n}{u_n} \bar{\delta}\Big(\sum_{i=1}^{k-1} \frac{u_i u_n}{\sigma_{in}} \langle ki \rangle - \langle kn \rangle \Big) \bar{\delta}\Big(\sum_{i=1}^{k} u_i u_n \langle ni \rangle \prod_{j \neq i}^{k} \sigma_{jn} \Big) J_k \right) , \end{aligned} \tag{5.26}$$

where J_k and J_{k-1} are the factors obtained by pulling out the products in the delta functions:

$$J_k = \prod_{j=1}^{k} \sigma_{jn}\,, \quad \text{and} \quad J_{k-1} = \prod_{q=k}^{n-1} \sigma_{qn}\,. \tag{5.27}$$

In conclusion, we have proved explicitly that the number of solutions for the refined scattering equations are the Eulerian numbers by using the method of dominance balance to establish the recursion relation. The results also give us the recursion relation for the reduced measure, where the two terms are localized on an $n-k-1$ and $k-1$ degree curve.

Chapter 6

Number of Solutions at One-Loop 1: CHY

Extending the RSVW formula to loop level is not so simple. It was proposed by Dolan and Goddard in [9] to use the genus g curve for a g loop amplitude, but this does not simply give us pure Yang-Mills [1]. The RSVW formula uses spinors in 4d, meaning that the formula is manifestly on-shell in 4d, thus it is not clear how one would incorporate the off-shell loop-momentum. In this project, two formalism that will generalize the RSVW formula to describe off-shell loop momentum will be discussed: the Cachazo, He, and Yuan (CHY) formalism, which will be the subject of this chapter, and the spinorial formalism which will be the subject of the next chapter.

6.1 CHY formalism

The motivation for CHY formalism is that we want a generalization of RSVW formula to general dimension. To begin with, we want to look at the general structure of the refined scattering equation.

The general structure of the refined scattering equations are the pole structure summing over $1/(\sigma - \sigma_i)$, and the spinors of the momentum at those marked points in the numerator. If we were to have a formula for arbitrary dimension, the first thing that need to be changed is the spinor, since the structure of the spinors would change significantly for the other dimensions. So we want the numerator to be some function of the momenta. The constraint equations that the CHY formalism proposed in [10] are the so-called scattering equations

$$\mathcal{E}_i = \sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_{ij}} = 0 \,. \tag{6.1}$$

[1]This is due to the fact that RSVW came from twistor string theory, in which the theory include the coupling to conformal gravity at the loop level. Although, this project will not be focusing on twistor string and conformal gravity

One can see that these satisfy the general structure that is required. Moreover, the scattering equations are implied by the refined scattering equations since

$$k_i \cdot \left(\sum_{j\neq i} \frac{\kappa_j^{\alpha}}{\sigma_{ij}} \tilde{\mathcal{E}}_j^{\dot\alpha} + \sum_{p} \frac{\tilde{\kappa}_p^{\dot\alpha}}{\sigma_{ip}} \mathcal{E}_p^{\alpha} \right) = \sum_{j=1\neq i}^{k} \frac{[ij]\,\langle ij\rangle}{\sigma_{ij}} + \sum_{p=k+1}^{k} \frac{[ip]\,\langle ip\rangle}{\sigma_{ip}} = \sum_{j\neq i} \frac{k_i \cdot k_j}{\sigma_{ij}} = 0\,, \quad (6.2)$$

on the support of the refined scattering equation for some momentum k_j with $j \in 1, ..., k$, and the same argument works for $j \in k+1, ..., n$. The number of solutions for the scattering equations are given in [21] as

$$\mathcal{N}^{(0)}_{\mathrm{CHY}}(n) = (n-3)!\,. \quad (6.3)$$

This can be proven by taking the soft limit of the n^{th} particle. The rest of the scattering equation drops the term containing k_n, reducing into scattering equations for $n-1$ particles. The n^{th} equation gives $(n-3)$ solutions for σ_n.

The formula for arbitrary dimension Yang-Mills scattering amplitude is presented as

$$\mathcal{A}_n = \int \frac{d^n\sigma}{\mathrm{vol}SL(2,\mathbb{C})} \prod_a{}' \delta\left(\sum_{b\neq a} \frac{k_a \cdot k_b}{\sigma_{ab}} \right) \frac{\mathcal{I}_n(\{k,\epsilon,\sigma\})}{\sigma_{12}...\sigma_{n1}}\,, \quad (6.4)$$

where

$$\prod_a{}' \delta\left(\sum_{b\neq a} \frac{k_a \cdot k_b}{\sigma_{ab}} \right) = \sigma_{ij}\sigma_{jk}\sigma_{ki} \prod_{a\neq i,j,k} \delta\left(\sum_{b\neq a} \frac{k_a \cdot k_b}{\sigma_{ab}} \right)\,,$$

from fixing three points on the $SL(2,\mathbb{C})$ invariant Riemann sphere and $\mathcal{I}_n(\{k,\epsilon,\sigma\})$ is a permutation invariant function of σ_a, momenta k_a^{μ} and polarization vectors ϵ_a^{μ}. Explicitly, it is the reduced Pfaffian[2] of a $2n \times 2n$ antisymmetric matrix

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}\,, \quad (6.5)$$

with

$$A_{ab} = \begin{cases} \frac{k_a \cdot k_b}{\sigma_{ab}} & a \neq b\,, \\ 0 & a = b\,, \end{cases} \qquad B_{ab} = \begin{cases} \frac{\epsilon_a \cdot \epsilon_b}{\sigma_{ab}} & a \neq b\,, \\ 0 & a = b\,, \end{cases} \qquad \text{and} \quad C_{ab} = \begin{cases} \frac{\epsilon_a \cdot k_b}{\sigma_{ab}} & a \neq b\,, \\ -\sum_{c\neq a} \frac{\epsilon_a \cdot k_c}{\sigma_{ac}} & a = b\,. \end{cases} \quad (6.6)$$

The Pfaffian of Ψ vanishes since the first n rows of Ψ are linearly independent. The reduced Pfaffian, however, does not vanish, and it is permutation invariant in the sense that it is independent of the choice of i and j. It was shown in [23] and [24] that the CHY formula in 4d is equivalent to the RSVW where the constraints and the reduced Pfaffian $\mathrm{Pf}'\,\Psi$ are equal to the RSVW constraints.

[2]Reduced Pfaffian is a Pfaffian of the matrix with two rows and columns removed. Explicitly, the reduced Pfaffian with rows i, j and columns i, j removed is defined as $\mathrm{Pf}'\,\Psi := 2\frac{(-1)^{i+j}}{\sigma_{ij}}\,\mathrm{Pf}(\Psi^{ij}_{ij})$.

The one loop formula for the Yang-Mill scattering amplitude in the CHY formalism was given in [11] as

$$\mathcal{A}_n^{(1)} = \int \frac{d\sigma_+ d\sigma_- d^n\sigma}{\text{vol}SL(2,\mathbb{C})} \prod_i{}' \delta\left(\mathcal{E}_i\right) \delta\left(\mathcal{E}_+\right) \delta\left(\mathcal{E}_-\right) \mathcal{I}_{loop}\,. \tag{6.7}$$

where the constraints are given by

$$\mathcal{E}_i = \Big(\sum_j \frac{k_i \cdot k_j}{\sigma_{ij}} + \frac{l \cdot k_i}{\sigma_{i+}} - \frac{l \cdot k_i}{\sigma_{i-}}\Big) \qquad \mathcal{E}_\pm = \pm\Big(\sum_j \frac{l \cdot k_j}{\sigma_{\pm j}}\Big)\,, \tag{6.8}$$

for loop momentum l. This formula proposed that the loop amplitude can be evaluated on a Riemann sphere with a node as illustrated in Fig. 6.1. The loop momentum will flow through the node and it can be viewed as outgoing with momentum $+l$ and incoming with momentum $-l$ at the nodal points denoted by σ_+ and σ_-, respectively.

The number of solutions to the one loop scattering equations was shown in [12]. Since the one loop scattering equations are just the tree level scattering equations with the loop part on the forward limit, direct counting using (6.3) with $n+2$ particles gives $(n-1)!$ solutions, but this is not the case. In fact, not all of the $(n-1)!$ are regular solutions; there are singular solutions that does not satisfies the loop scattering equation in the forward limit. This can be proposed as the following:

Proposition 1. The number of regular solutions for the one loop scattering equations are given by

$$\mathcal{N}^{(1)}(n) = (n-1)! - (n-2)!\,, \tag{6.9}$$

where the $(n-2)!$ are the singular solutions that is excluded from the forward limit.

In this chapter, we will proof proposition 1 in two different ways

1. using the same method in [12] and [13] with some improvement, and
2. using the method of dominance balance to construct the recursion relation for finding the number of solutions.

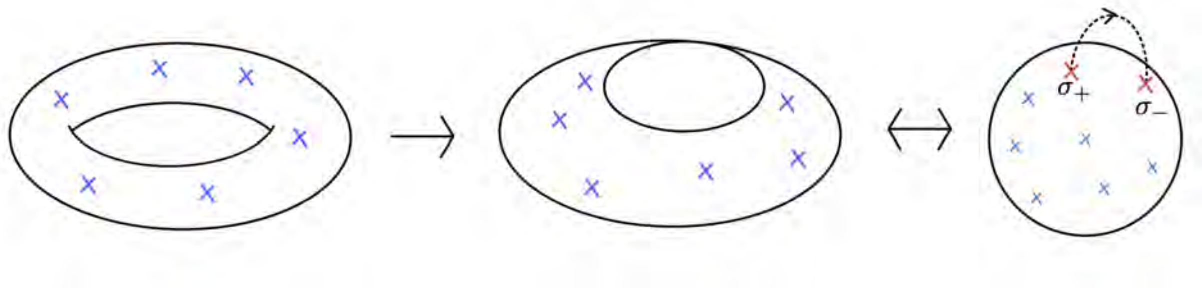


Figure 6.1: The deformation of a torus with 6 marked points into a nodal Riemann sphere with 6 + 2 marked points, where the extra marked points are from the nodal points.

6.2 Dimensional Reduction and Back-to-Back limit of the Scattering Equations

In this section, we will review the proof of proposition 1 that was presented in [12] and [13] in details with some additional improvement for completion. It was proposed that the one loop scattering equations can be understand as a special version of tree level scattering equation embedded in higher dimension. We can achieve this by embedding the momentum k in 4d into higher dimension, denoted K, and requiring them to satisfy some properties:

1. All momenta are on-shell in higher dimension:
$$K_A^2 = 0 \qquad \forall\, A \in \{1, .., n, n+1, n+2\}\,, \tag{6.10}$$
where $n+1$ and $n+2$ are just convenience labels for the loop momenta embedded in higher dimension. For the 4d on-shell momenta, this follows directly from trivially embedding them by
$$K_i = (k_i, 0) \qquad \forall\, i, \in \{1, .., n\} \tag{6.11}$$
2. The dot products of the 4d on-shell momentum is preserved under the embedding
$$K_i \cdot K_j = k_i \cdot k_j \qquad \forall\, i, j \in \{1, .., n\}\,. \tag{6.12}$$
3. The scattering equations in the higher dimension are
$$\mathcal{E}_A = \sum_{B=1}^{n+2} \frac{K_A \cdot K_B}{\sigma_{AB}}\,. \tag{6.13}$$
4. The parameter τ governs the back-to-back limit. We can take the forward limit such that the loop momenta at the two nodal points are back-to-back, i.e. $k_{n+1} = l$ and $k_{n+2} = -l$, by taking the limit $\tau \to 0$. In this limit, (6.13) become (6.8).

For the 4d off-shell momenta, the embedding with the lowest number of extra dimension that satisfies property 1 and 2 is
$$K_{n+1} = (l + \tau q, |l| + \tau|q|, \tau p, i\tau p) \tag{6.14}$$
$$K_{n+2} = (-l + \tau q, -|l| + \tau|q|, \tau p, -i\tau p) \tag{6.15}$$
and this can be observed that
$$K_{n+1} \cdot K_{n+2} = -2\tau^2 p^2 \tag{6.16}$$
showing that the rate of back-to-back limit grows as τ^2. The fact that this is of the order τ^2 gaurantees that the l^2 will vanish on the back-to-back limit. Substituting the expression for K_A's into the scattering equation (6.13),
$$\mathcal{E}_i^{(\tau)} = \Big(\sum_j \frac{k_i \cdot k_j}{\sigma_{ij}} + \frac{l \cdot k_i}{\sigma_{i+}} - \frac{l \cdot k_i}{\sigma_{i-}}\Big) + \tau\Big(\frac{q \cdot k_i}{\sigma_{i+}} + \frac{q \cdot k_i}{\sigma_{i-}}\Big) \tag{6.17}$$
$$\mathcal{E}_+^{(\tau)} = \Big(\sum_j \frac{l \cdot k_j}{\sigma_{+j}}\Big) + \tau\Big(\sum_j \frac{q \cdot k_j}{\sigma_{+j}}\Big) + \frac{1}{\sigma_{+-}}(-2\tau^2 p^2) \tag{6.18}$$
$$\mathcal{E}_-^{(\tau)} = \Big(-\sum_j \frac{l \cdot k_j}{\sigma_{-j}}\Big) + \tau\Big(\sum_j \frac{q \cdot k_j}{\sigma_{-j}}\Big) + \frac{1}{\sigma_{-+}}(-2\tau^2 p^2) \tag{6.19}$$

where the superscript of τ indicates that this version of the scattering equations still depends on the parameter τ, and will reduce to the one loop scattering equations (6.8) under the limit $\tau \to 0$.

Now that we have successfully embedded the 4d momenta into higher dimension satisfying the first two conditions given above. We will prove proposition 1 by showing that not all solutions of (6.13) satisfies (6.8) on the back-to-back limit.

Proof. On the limit $\tau \to 0$, the higher dimension scattering equations (6.17) drop all the term with τ, giving directly the one loop scattering equations (6.8). However, some of the $(n-1)!$ solutions of (6.13) do not satisfy (6.8). We need to investigate each cases of solutions, which are the two singular solutions and the regular solutions.

Singular Solutions We can consider the case that $\sigma_- = \sigma_+ + \epsilon$ for some $\epsilon \ll 1$. This make the τ-dependent scattering equations (6.19) become

$$\begin{aligned}\mathcal{E}_i^{(\tau)} &= \Big(\sum_j \frac{k_i \cdot k_j}{\sigma_{ij}} + \frac{l \cdot k_i}{\sigma_{i+}} - \frac{l \cdot k_i}{\sigma_{i+} - \epsilon}\Big) + \tau\Big(\frac{q \cdot k_i}{\sigma_{i+}} + \frac{q \cdot k_i}{\sigma_{i+} - \epsilon}\Big) \\ &= \Big(\sum_j \frac{k_i \cdot k_j}{\sigma_{ij}} - \epsilon\frac{l \cdot k_i}{\sigma_{i+}^2}\Big) + \tau\Big(\frac{2q \cdot k_i}{\sigma_{i+}}\Big) + \tau\epsilon\Big(\frac{q \cdot k_i}{\sigma_{i+}^2}\Big) \end{aligned} \tag{6.20}$$

$$\mathcal{E}_+^{(\tau)} = \Big(\sum_j \frac{l \cdot k_j}{\sigma_{+j}}\Big) + \tau\Big(\sum_j \frac{q \cdot k_j}{\sigma_{+j}}\Big) + \frac{2\tau^2 p^2}{\epsilon} \tag{6.21}$$

$$\begin{aligned}\mathcal{E}_-^{(\tau)} &= \Big(-\sum_j \frac{l \cdot k_j}{\sigma_{+j} + \epsilon}\Big) + \tau\Big(\sum_j \frac{q \cdot k_j}{\sigma_{+j} + \epsilon}\Big) - \frac{2\tau^2 p^2}{\epsilon} \\ &= -\Big(\sum_j \frac{l \cdot k_j}{\sigma_{+j}} - \epsilon\frac{l \cdot k_j}{\sigma_{+j}^2}\Big) + \tau\Big(\sum_j \frac{q \cdot k_j}{\sigma_{+j}} - \epsilon\frac{q \cdot k_j}{\sigma_{+j}^2}\Big) - \frac{2\tau^2 p^2}{\epsilon}, \end{aligned} \tag{6.22}$$

Since both ϵ and τ are small, the term of order $\tau\epsilon$ can be safely dropped;

$$\mathcal{E}_i^{(\tau)} = \Big(\sum_j \frac{k_i \cdot k_j}{\sigma_{ij}} - \epsilon\frac{l \cdot k_i}{\sigma_{i+}^2}\Big) + \tau\Big(\frac{2q \cdot k_i}{\sigma_{i+}}\Big) \tag{6.23}$$

$$\mathcal{E}_+^{(\tau)} = \Big(\sum_j \frac{l \cdot k_j}{\sigma_{+j}}\Big) + \tau\Big(\sum_j \frac{q \cdot k_j}{\sigma_{+j}}\Big) + \frac{2\tau^2 p^2}{\epsilon} \tag{6.24}$$

$$\mathcal{E}_-^{(\tau)} = -\Big(\sum_j \frac{l \cdot k_j}{\sigma_{+j}} - \epsilon\frac{l \cdot k_j}{\sigma_{+j}^2}\Big) + \tau\Big(\sum_j \frac{q \cdot k_j}{\sigma_{+j}}\Big) - \frac{2\tau^2 p^2}{\epsilon}. \tag{6.25}$$

The possible balances that can be considered are the two cases where $\epsilon \sim \tau$ and $\epsilon \sim \tau^2$.

Singular Solution I Let us consider the case where $\epsilon \sim \tau$. One can see that within this choice of dominance balance, the terms $2\tau^2 p^2/\epsilon$ vanish nicely on the limit $\tau \to 0$.

The other terms of order $\mathcal{O}(\epsilon)$ also vanish nicely. On this limit, the scattering equations for the i-th particles reduce to the usual scattering equation without loop-terms

$$\mathcal{E}_i = \sum_j \frac{k_i \cdot k_j}{\sigma_{ij}}\,, \tag{6.26}$$

and these equations have $(n-3)!$ solutions. The solution to the plus and minus scattering equations can be determined by considering the sum and difference of them:

$$\mathcal{E}_+^{(\tau)} + \mathcal{E}_-^{(\tau)} = 2\tau\Big(\sum_j \frac{q \cdot k_j}{\sigma_{+j}}\Big) + \epsilon\Big(\sum_j \frac{l \cdot k_j}{\sigma_{+j}^2}\Big) \tag{6.27}$$

$$\mathcal{E}_+^{(\tau)} - \mathcal{E}_-^{(\tau)} = \frac{4\tau^2 p^2}{\epsilon} - 2\Big(\sum_j \frac{l \cdot k_j}{\sigma_{+j}}\Big) - \epsilon\Big(\sum_j \frac{l \cdot k_j}{\sigma_{+j}^2}\Big). \tag{6.28}$$

The first equation (6.27) gives a solution for ϵ (which is $\epsilon \sim \tau$) and the second equation, on the limit $\tau \to 0$ yields an $(n-2)$ degree polynomial in σ_+:

$$\sum_{j=1}^{n} \frac{l \cdot k_j}{\sigma_{+j}} = 0\,. \tag{6.29}$$

Note that naively this looks like an $(n-1)$ degree polynomial, the degree is reduced due to the fact that the leading order vanishes by the momentum conservation

$$\sum_{j=1}^{n} (l \cdot k_j)\sigma_+^{n-1} = 0. \tag{6.30}$$

Therefore, there are $(n-3)! \times (n-2) \times 1 = (n-2)!$ solutions for this choice of dominance balance.

Singular Solution II For the second choice of dominance balance, $\epsilon \sim \tau^2$, the equations for i-th particles reduce in the limit $\tau \to 0$ to the normal scattering equations, giving $(n-3)!$ solutions. One can see that the terms $2\tau^2 p^2/\epsilon$ do not vanish in this case, so it contributes as an extra term to the equations. This is not the correct one loop scattering equations therefore the solutions would not solve the loop scattering equations either.
In order to count the number of solutions for this case, one can see that (6.28) gives one solution for ϵ (which is $\epsilon \sim \tau^2$) and the term of order $\mathcal{O}(\epsilon)$ in (6.27) vanishes, leaving the equation

$$\sum_{j=1}^{n} \frac{q \cdot k_j}{\sigma_{+j}} = 0\,, \tag{6.31}$$

to be solved. This equation is also an $(n-2)$ degree polynomial in σ_+. Therefore, there are also $(n-3)! \times (n-2) \times 1 = (n-2)!$ solutions for this choice of dominance balance.

Regular Solutions The number of regular solutions to the loop scattering equations under the reduction from tree-level higher dimension and back-to-back limit must exclude the two sectors of the singular solutions. The number of regular solutions is therefore

$(n-1)! - 2(n-2)!$.

The solutions for the loop scattering equations are then the regular solutions and the first group of singular solutions, giving the total number of solutions

$$\mathcal{N}^{(1)}(n) = (n-1)! - (n-2)! \tag{6.32}$$

□

6.3 Soft Recursion of the One-Loop Scattering Equations

In this section, we will proof proposition 1 by establishing a recursion relation for the number of solutions to the one loop scattering equations. The recursion relation is obtained by using the method of dominance balance, i.e. taking the soft limit then solve for the recursion of each balances.

The number of solutions in (6.9) satisfy a recursion relation of the form

$$\begin{aligned}\mathcal{N}^{(1)}(n) &= (n-1)(n-2)! - (n-2)! = (n-1)(n-2)! - (n-2)(n-3)! \\ &= (n-1)[(n-2)! - (n-3)!] + (n-3)! \\ &= (n-1)\mathcal{N}^{(1)}(n-1) + (n-3)\mathcal{N}^{(0)}(n-1)\,,\end{aligned} \tag{6.33}$$

where $\mathcal{N}^{(0)}(n) = (n-3)!$ is the number of solutions for the n particle tree-level scattering equations. This recursion relation will be useful for the following proof.

Proof. Now, to find the recursion relation for the number of solutions, let us consider the soft limit of the loop scattering equation by taking $k_n \to \tau k_n$. This make the one loop scattering equations become

$$\begin{aligned}\mathcal{E}_i &= \sum_{j\neq n} \frac{k_i \cdot k_j}{\sigma_{ij}} + \frac{l \cdot k_i}{\sigma_{i+}} - \frac{l \cdot k_i}{\sigma_{i-}} + \tau \frac{k_i \cdot k_n}{\sigma_{in}} \\ \mathcal{E}_\pm &= \pm\Big(\sum_{j\neq n} \frac{l \cdot k_j}{\sigma_{\pm j}} + \tau \frac{l \cdot k_n}{\sigma_{\pm n}}\Big) \\ \mathcal{E}_n &= \tau\Big(\sum_{j\neq n} \frac{k_n \cdot k_j}{\sigma_{nj}} + \frac{l \cdot k_n}{\sigma_{n+}} - \frac{l \cdot k_n}{\sigma_{n-}}\Big)\,, .\end{aligned} \tag{6.34}$$

Now, we can consider the two dominance balances in the limit $\tau \to 0$: the first balance with all the terms in $\mathcal{E}_n$ are of order τ and the second balance where the marked points σ_n, σ_+, and σ_- are degenerated.

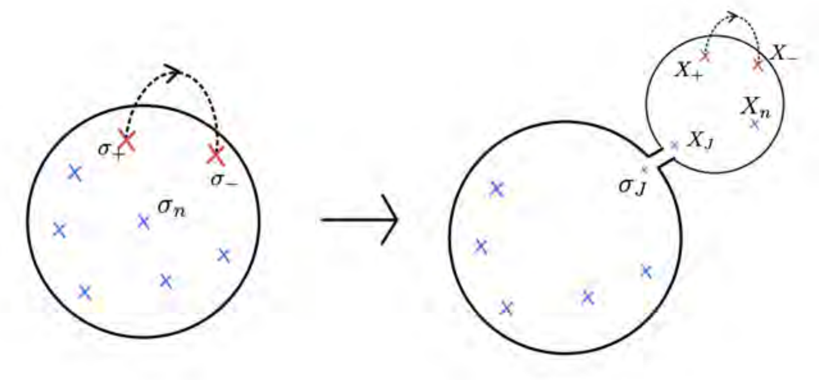


Figure 6.2: The geometric interpretation of the reparametrization in (6.38) as the subsphere X bubbling-off from the main Riemann sphere σ. Under the $SL(2,\mathbb{C})$ invariance, we can gauge fix three marked points on each sphere.

Balance 1 By taking the limit $\tau \to 0$, all scattering equations, except $\mathcal{E}_n$, reduce to the loop scattering equations for $(n-1)$ particles. In this dominance balance, we let all the terms in $\mathcal{E}_n$ have the same order. This allow us to express $\mathcal{E}_n$ as an $n-1$ degree polynomial in σ_n,

$$\sum_{j\neq n} k_n \cdot k_j \prod_{\substack{A=1,\pm\\ A\neq j}}^{n-1} \sigma_{nA} + l \cdot k_n \prod_{A=1,-}^{n-1} \sigma_{nA} - l \cdot k_n \prod_{A=1,+}^{n-1} \sigma_{nA} = 0\,, \tag{6.35}$$

where the $\pm$ in the products indicate the $\sigma_\pm$. This polynomial looks naively like a degree n polynomial, but the leading order vanishes since the coefficient of σ_n^n is $\sum k_i \cdot k_n + l - l = 0$, making it a degree $n-1$ polynomial in σ_n. Thus, there are $n-1$ solutions for σ_n in this dominance balance, hence we pick up the term

$$(n-1)\mathcal{N}^{(1)}(n-1)\,. \tag{6.36}$$

Balance 2 The other dominance balance is more difficult. It is achieved by reparametrizing σ_n and $\sigma_\pm$ as

$$\sigma_\pm = \sigma_J + \varepsilon X_\pm \tag{6.37}$$
$$\sigma_n = \sigma_J + \varepsilon X_n \tag{6.38}$$

where σ_J is the junction point and X's are the coordinates on the subsphere. This reparametrization has a geometric interpretation as the bubbling-off of the Riemann sphere into a sub-sphere, as illustrate in Fig. 6.2. Substituting the reparametrized σ_n and $\sigma_\pm$ into equations (6.34), give the relations

$$\mathcal{E}_\pm = \pm\Big(\sum_{j\neq n} \frac{l\cdot k_j}{\sigma_{Jj}}\Big(1 - \varepsilon\frac{X_\pm}{\sigma_{Jj}} + \varepsilon^2\frac{X_\pm^2}{\sigma_{Jj}^2}\Big) + \frac{\tau}{\varepsilon}\frac{l\cdot k_n}{X_{\pm n}}\Big) + \mathcal{O}(\varepsilon^3)$$
$$\mathcal{E}_n = \tau\sum_{j\neq n} \frac{k_n\cdot k_j}{\sigma_{Jj}}\Big(1 - \varepsilon\frac{X_n}{\sigma_{Jj}} + \varepsilon^2\frac{X_n^2}{\sigma_{Jj}^2}\Big) + \frac{\tau}{\varepsilon}\Big(\frac{l\cdot k_n}{X_{n+}} - \frac{l\cdot k_n}{X_{n-}}\Big) + \mathcal{O}(\varepsilon^3), \tag{6.39}$$

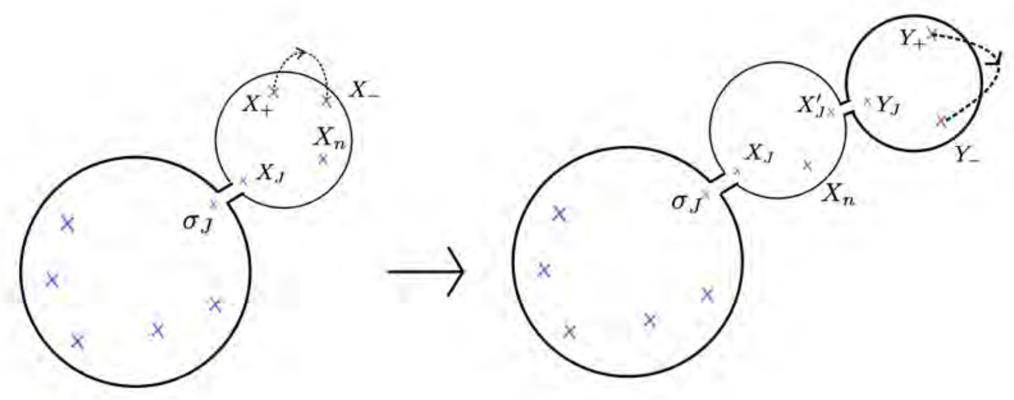


Figure 6.3: The geometric interpretation of the reparametrization in (6.40) as the subsphere Y bubbling-off from the sphere X.

telling that $X_+ = X_-$ at leading order. The parametrization must then include another order

$$\sigma_\pm = \sigma_J + \varepsilon X_+ + \hat{\varepsilon} Y_\pm \,. \tag{6.40}$$

where we are allowed to do a gauge fixing to set $Y_+ = 0$. This modification correspond to the point σ_+ and σ_- bubbling off into another subsphere as illustrated in fig 6.3. This make the expressions in (6.39) become

$$\begin{aligned}
\mathcal{E}_- &= -\sum_{j\neq n} \frac{l\cdot k_j}{\sigma_{Jj}}\Big(1 - \varepsilon\frac{(X_+ + \hat{\varepsilon}Y_-)}{\sigma_{Jj}} + \varepsilon^2\frac{(X_+ + \hat{\varepsilon}Y_-)^2}{\sigma_{Jj}^2}\Big) - \frac{\tau}{\varepsilon}\frac{l\cdot k_n}{X_{+n}}\Big(1 - \frac{\hat{\varepsilon}Y_-}{X_{+n}} + \frac{\hat{\varepsilon}^2 Y_-^2}{X_{+n}^2}\Big) + \mathcal{O}(\varepsilon^3)\\
\mathcal{E}_+ &= \Big(\sum_{j\neq n}\frac{l\cdot k_j}{\sigma_{Jj}}\Big(1 - \varepsilon\frac{X_+}{\sigma_{Jj}} + \varepsilon^2\frac{X_+^2}{\sigma_{Jj}^2}\Big) + \frac{\tau}{\varepsilon}\frac{l\cdot k_n}{X_{+n}}\Big) + \mathcal{O}(\varepsilon^3)\\
\mathcal{E}_n &= \tau\sum_{j\neq n}\frac{k_n\cdot k_j}{\sigma_{Jj}} + \frac{\tau\hat{\varepsilon}}{\varepsilon}\Big(\frac{l\cdot k_n Y_-}{X_{n+}^2}\Big) + \mathcal{O}(\varepsilon).
\end{aligned} \tag{6.41}$$

A short calculation can be done to show that

$$\begin{aligned}
\mathcal{E}_- = -\Bigg(&\sum_{j\neq n}\frac{l\cdot k_j}{\sigma_{Jj}} - \sum_{j\neq n}\frac{l\cdot k_j}{\sigma_{Jj}^2}\varepsilon X_+ + \sum_{j\neq n}\frac{l\cdot k_j}{\sigma_{Jj}^3}\varepsilon^2 X_+^2 + \frac{\tau}{\varepsilon}\frac{l\cdot k_n}{X_{+n}} + 2\sum_{j\neq n}\frac{l\cdot k_j}{\sigma_{Jj}^3}\varepsilon^2\hat{\varepsilon}X_+Y_-\\
&+\sum_{j\neq n}\frac{l\cdot k_j}{\sigma_{Jj}^3}\varepsilon^2\hat{\varepsilon}^2 Y_-^2 - \sum_{j\neq n}\frac{l\cdot k_j}{\sigma_{Jj}^2}\varepsilon\hat{\varepsilon}Y_- - \frac{\tau\hat{\varepsilon}}{\varepsilon}\frac{l\cdot k_n Y_-}{X_{+n}^2} + \frac{\tau\hat{\varepsilon}^2}{\varepsilon}\frac{l\cdot k_n Y_-^2}{X_{+n}^3}\Bigg)\,.
\end{aligned} \tag{6.42}$$

On the back-to-back limit, we expect that $\mathcal{E}_+$ and $\mathcal{E}_-$ are also back-to-back i.e. $\mathcal{E}_- \sim -\mathcal{E}_+$. The first four terms of $\mathcal{E}_-$ give the correct back-to-back expression, so we require that the rest of the terms that are absent in $\mathcal{E}_+$ to vanish. This can be achieved by imposing that $\varepsilon\hat{\varepsilon}$ and $\tau\hat{\varepsilon}/\varepsilon$ are of the same order, so that $\varepsilon \sim \pm\tau^{1/2}$.
Using the soft limit, $\mathcal{E}_+$ gives

$$\mathcal{E}_+ = \sum_{j\neq n}\frac{l\cdot k_j}{\sigma_{Jj}}\,, \tag{6.43}$$

which is an $(n-3)$ degree polynomial in σ_J, giving exactly the $n-3$ solutions for σ_J, and $\mathcal{E}_n$ imposes $\hat{\varepsilon} \sim \varepsilon$. Therefore, this dominance balance pick up the term

$$(n-3)\mathcal{N}^{(0)}(n-1)\,. \tag{6.44}$$

Balance 3 However, if we suppose that the term $\tau\hat{\varepsilon}/\varepsilon$ is sub-leading instead, we must have $\varepsilon \sim \pm\tau^x$ for $x < 1/2$. Then, in $\mathcal{E}_+$ we would get the same $(n-3)$ degree polynomial in σ_J, and in $\mathcal{E}_-$ the leading order vanish by the conditions obtained from $\mathcal{E}_+$. The sub-leading of $\mathcal{E}_-$ now goes as

$$0 = \sum_{j\neq n} \frac{l\cdot k_j}{\sigma_{Jj}^2}\,, \tag{6.45}$$

which is certainly a different equation, hence would give contradicting solutions to the $n-3$ degree polynomial in σ_n obtained from $\mathcal{E}_+$. Therefore this is not a dominance balance.
The recursion relation for the number of solutions to the one loop scattering equation with n external particles is therefore

$$\mathcal{N}^{(1)}(n) = (n-1)\mathcal{N}^{(1)}(n-1) + (n-3)\mathcal{N}^{(0)}(n-1)\,. \tag{6.46}$$

Since the initial value of this recursion is $\mathcal{N}^{(1)}(4) = 4$, similar to the the initial value of (6.9), they are the same recursion relation. □

We will see in the next chapter that some of the feature here, especially the second balance, will reappear in the spinorial version.

Chapter 7

Number of Solutions at One-Loop 2: Loop Polarized Scattering Equations

The other method of calculating one loop amplitude that will be the subject of this project is the spinor formalism. This formula for the one-loop superamplitude was recently proposed in [14], and it sparks our curiosity on its number of solutions and its difference or similarity it to the number of solutions from the CHY formalism. The formula for one loop superamplitude was proposed in [14] as

$$\mathcal{A}^{(1)}_{4\text{D},n} = \int \frac{d^4 l}{l^2} \sum_{\text{states}} \mathcal{A}^{(0)}_{\text{off-shell},n+2}\Big|_{F.L.}, \tag{7.1}$$

where $\mathcal{A}^{(1)}_{4\text{D},n}$ denotes the n-point one loop amplitude in 4d and $\mathcal{A}^{(0)}_{\text{off-shell},n+2}\Big|_{F.L.}$ denotes the off-shell $(n+2)$ point tree-level amplitude, where the $+2$ are the off-shell momenta, on the forward limit[1]. The details of this formula will be discussed in the following section.

The off-shell superamplitude was obtained using a 6d twistor formula, and it is localized on a set of constraint equations called the polarized scattering equation ([25], [26]). In the following section, a brief introduction to the polarized scattering equation will be presented, followed by the construction of the one loop amplitude of [14]. Then, we show the progress on establishing the recursion relation for the number of solutions of the loop polarized scattering equation, using the method of dominance balance.

The results were unexpected to us. To this point, we have found two dominance balances for the loop polarized scattering equations. The first part give a result that looks like the number of solutions from CHY formalism, and the other part give a result that looks like the number of solutions from RSVW formula. This is interesting because the CHY formalism is independent of the MHV degree, but RSVW formula depends on the MHV degree.[2]

[1] In supersymmetric theory, the sum over states is generalized into Grassmannian integrals. In this project, we are mostly interested in the constraints, so the Grassmannian integrals can be set aside.

[2] The number of solutions for the tree polarized scattering equations is also independent of the MHV degree, given by $(n-3)!$.

7.1 Polarized Scattering Equations

In light of [26], the loop polarized scattering equation can be obtained by looking at the scattering equation with 6d spinors. For simplicity, we embedded the 6d spinors into 4d and works with them instead.[3] In order to see what off-shell momenta looks like in the spinor-helicity formalism, we can first notice that the off-shell momentum l can be written as a sum over two null momenta: $l = p_1 + p_2$ and $p_1^2 = 0 = p_2^2$. In the spinor-helicity formalism , the two null momenta can be written as two spinors. For convenience, we label them as

$$p_1 = \kappa_1^\alpha \tilde{\kappa}_2^{\dot{\alpha}} \quad \text{and} \quad p_2 = -\kappa_2^\alpha \tilde{\kappa}_1^{\dot{\alpha}} \,. \tag{7.2}$$

In this parametrization, the indices $1, 2$ are the label for the 4d massive little group $SL(2,\mathbb{C})$. The group $SL(2,\mathbb{C})$ has a natural metric ϵ_{ab}, so we can write the off-shell momentum as the antisymmetric product of the spinors with two indices:

$$l_{\alpha\dot{\alpha}} := \kappa_\alpha^a \tilde{\kappa}_{\dot{\alpha}}^b \epsilon_{ab} \,, \tag{7.3}$$

with

$$\kappa_\alpha^a \kappa_\beta^b \epsilon_{ab} := M\epsilon_{\alpha\beta} \quad \text{and} \quad \tilde{\kappa}_{\dot{\alpha}}^a \tilde{\kappa}_{\dot{\beta}}^b \epsilon_{ab} := \tilde{M}\epsilon_{\dot{\alpha}\dot{\beta}} \,, \tag{7.4}$$

so that $M\tilde{M} = l^2$. We denote the inner product of the little group index with the double brackets: $v_\alpha^a w_{\dot{\alpha}}^b \epsilon_{ab} := (\!(v_\alpha w_{\dot{\alpha}})\!)$. The spinors with two indices are then defined as

$$\kappa_\alpha^a = \left(\kappa_\alpha^1, \kappa_\alpha^2\right) \quad \text{and} \quad \tilde{\kappa}_{\dot{\alpha}}^a = \left(\tilde{\kappa}_{\dot{\alpha}}^1, \tilde{\kappa}_{\dot{\alpha}}^2\right) \,. \tag{7.5}$$

This notation allow us to embed the massless momenta in the new spinor notation as

$$\kappa_\alpha^a = (0, \kappa_\alpha) \quad \text{and} \quad \tilde{\kappa}_{\dot{\alpha}}^a = (\tilde{\kappa}_{\dot{\alpha}}, 0) \,, \tag{7.6}$$

so that we have the usual spinor-helicity decomposition: $(\!(\kappa_\alpha \tilde{\kappa}_{\dot{\alpha}})\!) = \kappa_\alpha \tilde{\kappa}_{\dot{\alpha}} = k_{\alpha\dot{\alpha}}$.
In [14], the forward limit of the loop momenta was proposed, and it can be written in this notation as

$$\kappa_{+,\alpha}^a = \left(\kappa_{+,\alpha}^1, \kappa_{+,\alpha}^2\right) \qquad \tilde{\kappa}_{+,\dot{\alpha}}^a = \left(\tilde{\kappa}_{+,\dot{\alpha}}^1, \tilde{\kappa}_{+,\dot{\alpha}}^2\right) \,, \tag{7.7}$$

$$\kappa_{-,\alpha}^a = \left(\kappa_{+,\alpha}^1, -\kappa_{+,\alpha}^2\right) \qquad \tilde{\kappa}_{-,\dot{\alpha}}^a = \left(\tilde{\kappa}_{+,\dot{\alpha}}^1, -\tilde{\kappa}_{+,\dot{\alpha}}^2\right) \,, \tag{7.8}$$

so that $l_{\alpha\dot{\alpha}} = (\!(\kappa_{+\alpha}\tilde{\kappa}_{+\dot{\alpha}})\!)$ and $-l_{\alpha\dot{\alpha}} = (\!(\kappa_{-\alpha}\tilde{\kappa}_{-\dot{\alpha}})\!) = -(\!(\kappa_{+\alpha}\tilde{\kappa}_{+\dot{\alpha}})\!)$.
The polarization can be defined as

$$\epsilon_{-a} = (0, 1) \quad \text{and} \quad \epsilon_{+a} = (1, 0) \,, \tag{7.9}$$

up to the reference choice. This allow us to write the product of the polarization and the spinors as

$$\epsilon_{i,\alpha} = ((\epsilon_- \kappa_{i,\alpha})) = \kappa_{i,\alpha} \qquad \tilde{\epsilon}_{i,\dot{\alpha}} = ((\epsilon_- \tilde{\kappa}_{i,\dot{\alpha}})) = 0 \qquad i \in - \,, \tag{7.10}$$

$$\epsilon_{p,\alpha} = ((\epsilon_+ \kappa_{p,\alpha})) = 0 \qquad \tilde{\epsilon}_{p,\dot{\alpha}} = ((\epsilon_+ \tilde{\kappa}_{p,\dot{\alpha}})) = \tilde{\kappa}_{p,\dot{\alpha}} \qquad p \in + \,, \tag{7.11}$$

$$\epsilon_{+,\alpha} = ((\epsilon_+ \kappa_{+,\alpha})) = \kappa_{+,\alpha}^1 \qquad \tilde{\epsilon}_{+,\dot{\alpha}} = ((\epsilon_+ \tilde{\kappa}_{+,\dot{\alpha}})) = \tilde{\kappa}_{+,\dot{\alpha}}^1 \qquad \text{for} + l \,, \tag{7.12}$$

$$\epsilon_{-,\alpha} = ((\epsilon_- \kappa_{-,\alpha})) = \kappa_{-,\alpha}^2 = -\kappa_{+,\alpha}^2 \qquad \tilde{\epsilon}_{-,\dot{\alpha}} = ((\epsilon_- \tilde{\kappa}_{-,\dot{\alpha}})) = \tilde{\kappa}_{-,\dot{\alpha}}^2 = -\tilde{\kappa}_{+,\dot{\alpha}}^2 \qquad \text{for} - l \,. \tag{7.13}$$

[3] See Appendix D for more details on the 6d spinors and the embedding.

This definition of the polarization allow us to write the polarization vectors for the massless momenta as $\epsilon_i + \tilde{\epsilon}_i$, and allow us to easily generalized them for the massive momenta.

Now that we have set-up the notation for the spinors, we can begin to introduce the polarized scattering equations. For tree level, the polarized scattering equations are given by

$$\mathcal{E}_{i\alpha} = u_{i,a}\lambda^a_\alpha(\sigma_i) - v_{i,a}\kappa^a_{i,\alpha} \quad \text{and} \quad \tilde{\mathcal{E}}_{i,\dot\alpha} = u_{i,a}\tilde{\lambda}^a_{\dot\alpha}(\sigma_i) - v_{i,a}\tilde{\kappa}^a_{i\dot\alpha}\,, \tag{7.14}$$

where there are now $4n$ equations and five variables $(\sigma_i, u_{i,a}, v_{i,a})$ associating to each particles and the map $\lambda^a_\alpha(\sigma)$ and $\tilde{\lambda}^a_{\dot\alpha}(\sigma)$ are given by

$$\lambda^a_\alpha(\sigma) = \sum_{i=1}^n \frac{u^a_i \epsilon_{i\alpha}}{\sigma - \sigma_i} \quad \text{and} \quad \tilde{\lambda}^a_{\dot\alpha}(\sigma) = \sum_{i=1}^n \frac{u^a_i \tilde{\epsilon}_{i\dot\alpha}}{\sigma - \sigma_i}\,. \tag{7.15}$$

The polarized scattering equations also imply the scattering equations, in fact, the conditions

$$u_{i,a}\lambda^a_\alpha(\sigma_i) = v_{i,a}\kappa^a_{i,\alpha} \quad \text{and} \quad u_{i,a}\tilde{\lambda}^a_{\dot\alpha}(\sigma_i) = v_{i,a}\tilde{\kappa}^a_{i\dot\alpha}\,, \tag{7.16}$$

are the relations obtained from the scattering equations. It presents linear relations between $(\lambda^a_\alpha(\sigma), \tilde{\lambda}^a_{\dot\alpha}(\sigma))$ and $(\kappa, \tilde{\kappa})$ using new scale-invariant variables u_i and v_i. From the scale invariance, we can normalize v_i using $\langle\!\langle v_i \epsilon_i \rangle\!\rangle = 1$.
The measure of the integral formula on the constraints of the polarized scattering equation and the normalization is given by

$$d\mu_n^{\text{pol}} := \frac{\prod_{i=1}^n d\sigma_i d^2u_i d^2v_i}{\text{vol}(SL(2,\mathbb{C})_\sigma \times SL(2,\mathbb{C}))} \prod_{i=1}^n \delta\left(\langle v_i \epsilon_i \rangle - 1\right) \delta^4\left(u_{ia}\lambda^a_A(\sigma) - v_{ia}\kappa^a_{iA}\right)\,, \tag{7.17}$$

where the quotients are over Riemann sphere's $SL(2,\mathbb{C})_\sigma$ and the little group's $SL(2,\mathbb{C})$. The integral formula for the scattering amplitude is then

$$A_n = \int d\mu_n^{\text{pol}} \mathcal{I}_n\,, \tag{7.18}$$

with

$$\mathcal{I}_n = \frac{\mathcal{I}_n^{\text{spin }1}}{\sigma_{12}\sigma_{23}\ldots\sigma_{n1}}\,, \tag{7.19}$$

where the $\mathcal{I}_n^{\text{spin }1}$ is a function of Maxwell polarization data. [4]

The loop polarized scattering equations are presented in [14], using the formula above to incorportate the off-shell momenta. Then, the forward limit of the off-shell momenta were taken and thus the formula presented above was obtained as

$$\mathcal{A}^{(1)}_{\text{4D},n} = \int \frac{d^4l}{l^2} \sum_{\text{states}} \mathcal{A}^{(0)}_{\text{off-shell},n+2}\Big|_{F.L.}\,. \tag{7.20}$$

where the two additional particles are the two legs of the off-shell loop momenta, with the forward limit (D.13).

[4] This could be extend to describe superamplitudes by including an extra supersymmetric factor, see [26]

The $n+2$ particles loop polarized scattering equations in the off-shell tree amplitude are then given by

$$\mathcal{E}_{\mathbf{i}\alpha} = u_{\mathbf{i},a}\lambda^a_\alpha(\sigma_{\mathbf{i}}) - v_{\mathbf{i}a}\kappa^a_{\mathbf{i},\alpha} \quad \text{and} \quad \tilde{\mathcal{E}}_{\mathbf{i},\dot\alpha} = u_{\mathbf{i},a}\tilde\lambda^a_{\dot\alpha}(\sigma_{\mathbf{i}}) - v_{\mathbf{i},a}\tilde\kappa^a_{\mathbf{i}\dot\alpha}\,, \tag{7.21}$$

where $\mathbf{i} \in \{1, 2, ..., n, +, -\}$. The loop polarized scattering equations (7.21) with $\lambda^a_\alpha(\sigma_A)$ and $\tilde\lambda^a_{\dot\alpha}(\sigma_A)$ written explicitly then take the form of

$$\mathcal{E}_{\mathbf{i}\alpha} = u_{\mathbf{i},a}\Big(\sum_{i=1}^{n}\frac{u^a_i\epsilon_{i\alpha}}{\sigma_{i\mathbf{i}}} + \frac{u^a_+\epsilon_{+\alpha}}{\sigma_{+\mathbf{i}}} + \frac{u^a_-\epsilon_{-\alpha}}{\sigma_{-\mathbf{i}}}\Big) - v_{\mathbf{i},a}\kappa^a_{\mathbf{i}\alpha} \tag{7.22}$$

$$\tilde{\mathcal{E}}_{\mathbf{i}\dot\alpha} = u_{\mathbf{i},a}\Big(\sum_{i=1}^{n}\frac{u^a_i\tilde\epsilon_{i\dot\alpha}}{\sigma_{i\mathbf{i}}} + \frac{u^a_+\tilde\epsilon_{+\dot\alpha}}{\sigma_{+\mathbf{i}}} + \frac{u^a_-\tilde\epsilon_{-\dot\alpha}}{\sigma_{-\mathbf{i}}}\Big) - v_{\mathbf{i}a}\tilde\kappa^a_{\mathbf{i}\dot\alpha}\,. \tag{7.23}$$

In the next sections, we will try to establish a recursion relation for the number of solutions of the loop polarized scattering equations. We will take the soft limit of the loop polarized scattering equations, then solve for the number of solutions on the dominance balances.

7.2 Soft Recursion of the Loop Polarized Scattering Equations

We will now put to use what we have done in the previous chapters. In chapter 5, we have calculated spinorial quantities to solve for the number of solutions for RSVW, while in chapter 6 we have experienced the calculation for one loop scattering equations. For the one loop polarized scattering equations, we expect it to show both features. Currently, we have found that there are two dominance balances: 1) similar to the balance in RSVW that all the terms in $\mathcal{E}_n$ and $\tilde{\mathcal{E}}_n$ are of the order τ and 2) similar to the second balance of the one loop CHY that was interpreted as the bubbling-off of subspheres.

In order to be prepared for the upcoming calculation, we ran a numerical calculation for the one loop polarized scattering equations with four particles, and it was found that there are two solutions for all MHV sectors. However, beyond four particles, solving numerically is too difficult for the standard algorithm, and hence the search was discontinued. Nevertheless, this result does not imply that the number of solutions are the same for all MHV sectors if we go beyond four particles. However, as will be described in the following section, the recursion relation for the loop polarized scattering equations seems to depend on the tree level amplitude in the second balance, hinting that it will depend on the MHV degree. This shows an interesting mix of an MHV indepedent CHY amplitude and MHV dependent spinor-helicity formalism.

The numerical results also hint another interesting feature. From supersymmetric Ward identity (see C), the amplitude in the all plus, all minus, one plus, and one minus sectors vanish at all loop levels. Since there exist solutions from the constraint equations, we expect that the other parts of the formula must vanish for these MHV sectors. We are currently investigating whether the integrand vanishes for these sectors.

Before diving into the calculation, we want to give a brief summary of the results at this stage.

7.2.1 Summary of the Results

The aim of the following calculation is that, to establish a recursion relation for the number of solutions to the loop polarized scattering equations, we consider the dominance balances of the constraints under the soft limit. At this point, we discuss two dominance balances. The first balance is achieved by having $u_n \to \tau \hat{u}_n$, so that all the terms in $\mathcal{E}_n$ and $\tilde{\mathcal{E}}_n$ are of the same order. This balance contributes $(n-1)\mathcal{N}^{(1)}_{\text{pol}}(n-1,k)$ solutions to the recursion relation, in parallel with the first balance of the loop CHY. The second balance is achieved by having $u_n \to \tau\hat{u}_n$, $u_n \to \tau\hat{u}_\pm$ along with the reparametrization $\sigma_n = \sigma_J + \tau X_n$ and $\sigma_\pm = \sigma_J + \tau X_\pm + \tau^2 Y_\pm$, also mirroring the second balance of the loop CHY, which can be interpreted geometrically as fig. 6.3. A polynomial in σ_J is obtained from this balance, but we have not concluded what the contribution to the recursion would look like. Combining the contributions of the two balances, we have now picked up two terms in the recursion relation:

$$\mathcal{N}^{(1)}_{\text{pol}}(n,k) \supset (n-1)\mathcal{N}^{(1)}_{\text{pol}}(n-1,k) + c(n)\mathcal{N}^{(0)}_{\text{pol}}(n-1,k) + \dots\,, \tag{7.24}$$

where the coefficient $c(n)$ of the second term is yet to be determined and we expect to find more terms from other dominance balances. While these two dominance balances closely resemble the dominance balances of the loop CHY in the last section, we expect that switching the helicity of the n^{th} and k^{th} particle will contribute two more dominance balances: one with all terms in the n^{th} equations of the same order and another one by reparametrizing the sphere.

7.2.2 Balance 1

In this balance, we will try to look at the loop polarized scattering equations under the limits $\kappa_{n\alpha} \to \tau\kappa_{n\alpha}$ and $\tilde{\kappa}_{n\dot{\alpha}} \to \tau\tilde{\kappa}_{n\dot{\alpha}}$, with $\tau \to 0$. On the limit, we will try to finding the number of solutions using the recursion. The polarized scattering equations are then

$$\mathcal{E}_{i\alpha} = u_{ia}\Big(\sum_{j=1,j\neq i}^{k} \frac{u^a_j\kappa_{j\alpha}}{\sigma_{ji}} + \frac{u^a_+\kappa^1_{+\alpha}}{\sigma_{+i}} + \frac{u^a_-\kappa^2_{-\alpha}}{\sigma_{-i}}\Big) - v_{i2}\kappa_{i\alpha} \tag{7.25}$$

$$\tilde{\mathcal{E}}_{i\dot{\alpha}} = u_{ia}\Big(\sum_{q=k+1}^{n-1} -\frac{u^a_q\tilde{\kappa}_{q\alpha}}{\sigma_{qi}} - \tau\frac{u^a_n\tilde{\kappa}_{n\alpha}}{\sigma_{ni}} + \frac{u^a_+\tilde{\kappa}^1_{+\alpha}}{\sigma_{+i}} + \frac{u^a_-\tilde{\kappa}^2_{-\alpha}}{\sigma_{-i}}\Big) - \tilde{\kappa}_{i\alpha} \tag{7.26}$$

$$\mathcal{E}_{n\alpha} = u_{na}\Big(\sum_{j=1}^{k} \frac{u^a_j\kappa_{j\alpha}}{\sigma_{jn}} + \frac{u^a_+\kappa^1_{+\alpha}}{\sigma_{+n}} + \frac{u^a_-\kappa^2_{-\alpha}}{\sigma_{-n}}\Big) + \tau\kappa_{n\alpha} \tag{7.27}$$

$$\tilde{\mathcal{E}}_{n\dot{\alpha}} = u_{na}\Big(\sum_{q=k+1}^{n-1} -\frac{u^a_q\tilde{\kappa}_{q\alpha}}{\sigma_{qn}} + \frac{u^a_+\tilde{\kappa}^1_{+\alpha}}{\sigma_{+n}} + \frac{u^a_-\tilde{\kappa}^2_{-\alpha}}{\sigma_{-n}}\Big) - \tau v_{n1}\tilde{\kappa}_{n\alpha} \tag{7.28}$$

$$\mathcal{E}_{\pm\alpha} = u_{\pm,a}\Big(\sum_{j=1}^{k} \frac{u^a_j\kappa_{j\alpha}}{\sigma_{j\pm}} + \frac{u^a_\mp\kappa^c_{\mp\alpha}}{\sigma_{\mp\pm}}\Big) - v_{\pm,a}\kappa^a_{pm\alpha} \tag{7.29}$$

$$\tilde{\mathcal{E}}_{\pm\dot{\alpha}} = u_{\pm,a}\Big(\sum_{q=k+1}^{n-1} -\frac{u^a_q\tilde{\kappa}_{q\alpha}}{\sigma_{q\pm}} - \tau\frac{u^a_n\tilde{\kappa}_{n\alpha}}{\sigma_{n\pm}} + \frac{u^a_\mp\tilde{\kappa}^c_{\mp\alpha}}{\sigma_{\mp\pm}}\Big) - v_{\pm,a}\tilde{\kappa}^a_{\pm\alpha}\,. \tag{7.30}$$

where $c = 1$ for κ_+ and $\tilde{\kappa}_+$, and $c = 2$ for κ_- and $\tilde{\kappa}_-$. Apart from the n^{th} particle's equations, the other loop polarized scattering equations reduce to those of $n-1$ external particles, similar to what happened to the refined scattering equations. We proceed with the same strategy by looking at the n^{th} particle's equations, a dominance balance can be achieved by having $u_n \to \tau \hat{u}_n$:

$$\mathcal{E}_{n\alpha} = \tau \hat{u}_{na} \Big(\sum_{j=1}^{k} \frac{u_j^a \kappa_{j\alpha}}{\sigma_{jn}} + \frac{u_+^a \kappa_{+\alpha}^1}{\sigma_{+n}} + \frac{u_-^a \kappa_{-\alpha}^2}{\sigma_{-n}} \Big) + \tau \kappa_{n\alpha} \,, \tag{7.31}$$

$$\tilde{\mathcal{E}}_{n\dot{\alpha}} = \tau \hat{u}_{n,a} \Big(\sum_{q=k+1}^{n-1} -\frac{u_q^a \tilde{\kappa}_{q\alpha}}{\sigma_{qn}} + \frac{u_+^a \tilde{\kappa}_{+\alpha}^1}{\sigma_{+n}} + \frac{u_-^a \tilde{\kappa}_{-\alpha}^2}{\sigma_{-n}} \Big) - \tau v_{n1} \tilde{\kappa}_{n\alpha} \,. \tag{7.32}$$

The two loop polarized scattering equations are actually for equations with four unknown variables: σ_n, $\hat{u}_n^1$, $\hat{u}_n^2$, and v_{n1}. Performing the same trick, we dot in κ_n and κ_1 to (7.31) and $\tilde{\kappa}_n$ and $\tilde{\kappa}_1$ to (7.32) to separate the four equations:

$$0 = \sum_{j=1}^{k} \frac{((\hat{u}_n u_j)) \langle \kappa_j \kappa_n \rangle}{\sigma_{jn}} + \frac{((\hat{u}_n u_+)) \langle \kappa_+^1 \kappa_n \rangle}{\sigma_{+n}} + \frac{((\hat{u}_n u_-)) \langle \kappa_-^2 \kappa_n \rangle}{\sigma_{-n}} \,, \tag{7.33}$$

$$0 = \sum_{j=2}^{k} \frac{((\hat{u}_n u_j)) \langle \kappa_j \kappa_1 \rangle}{\sigma_{jn}} + \frac{((\hat{u}_n u_+)) \langle \kappa_+^1 \kappa_1 \rangle}{\sigma_{+n}} + \frac{((\hat{u}_n u_-)) \langle \kappa_-^2 \kappa_1 \rangle}{\sigma_{-n}} + \langle \kappa_n \kappa_1 \rangle \,, \tag{7.34}$$

$$0 = \sum_{q=k+1}^{n-1} -\frac{((\hat{u}_n u_q)) [\tilde{\kappa}_q \tilde{\kappa}_n]}{\sigma_{qn}} + \frac{((\hat{u}_n u_+)) [\tilde{\kappa}_+^1 \tilde{\kappa}_n]}{\sigma_{+n}} + \frac{((\hat{u}_n u_-)) [\tilde{\kappa}_-^2 \tilde{\kappa}_n]}{\sigma_{-n}} \,, \tag{7.35}$$

$$0 = \sum_{q=k+1}^{n-1} -\frac{((\hat{u}_n u_q)) [\tilde{\kappa}_q \tilde{\kappa}_1]}{\sigma_{qn}} + \frac{((\hat{u}_n u_+)) [\tilde{\kappa}_+^1 \tilde{\kappa}_1]}{\sigma_{+n}} + \frac{((\hat{u}_n u_-)) [\tilde{\kappa}_-^2 \tilde{\kappa}_1]}{\sigma_{-n}} - v_{n1} [\tilde{\kappa}_n \tilde{\kappa}_1] \,. \tag{7.36}$$

The last equation, (7.36), can be used to solve for v_{n1}. By looking at the first equation of the four, (7.33), and expanding the contraction of the little group index explicitly, one can see that

$$0 = \sum_{j=1,\pm}^{k} (\hat{u}_n^1 u_j^2 - \hat{u}_n^2 u_j^1) \frac{\langle jn \rangle}{\sigma_{jn}} \to \hat{u}_n^1 = \hat{u}_n^2 \frac{\Big(\sum_{j=1,\pm}^{k} u_j^1 \frac{\langle jn \rangle}{\sigma_{jn}} \Big)}{\Big(\sum_{j=1,\pm}^{k} u_j^2 \frac{\langle jn \rangle}{\sigma_{jn}} \Big)} \,, \tag{7.37}$$

giving a relation between $\hat{u}_n^1$ and $\hat{u}_n^2$. We can obtain a relation similar to this by using the tilded equation (7.35):

$$\hat{u}_n^1 = \hat{u}_n^2 \frac{\Big(\sum_{q=k+1}^{n-1} u_q^2 \frac{[qn]}{\sigma_{qn}} - u_+^2 \frac{[+^1 n]}{\sigma_{+n}} - u_-^2 \frac{[-^2 n]}{\sigma_{-n}} \Big)}{\Big(\sum_{q=k+1}^{n-1} u_q^1 \frac{[qn]}{\sigma_{qn}} - u_+^1 \frac{[+^1 n]}{\sigma_{+n}} - u_-^1 \frac{[-^2 n]}{\sigma_{-n}} \Big)} \,, \tag{7.38}$$

both can be used to solve for σ_n in the next process.[5]
Now, we can solve for σ_n by substituting (7.37) in (7.35) and by noticing that

$$u_q^2 \frac{\sum_{j=1,\pm}^{k} u_j^1 \frac{\langle jn \rangle}{\sigma_{jn}}}{\sum_{j=1,\pm}^{k} u_j^2 \frac{\langle jn \rangle}{\sigma_{jn}}} - u_q^1 = \sum_{j=1,\pm}^{k} \frac{\langle jn \rangle}{\sigma_{jn}} ((u_q u_j)) \,. \tag{7.39}$$

[5] Note that in these relations, if one of the component of $\hat{u}_n$ vanishes then the σ_n obtained in (7.38) will not coincide with the one from (7.37).

These allow us to simplify (7.35) as [6]

$$0=\sum_{i=1,\pm}^{k}\sum_{q=k+1,-\pm}^{n-1}((u_ju_q))\frac{\langle jn\rangle[qn]}{\sigma_{jn}\sigma_{qn}}\,. \tag{7.40}$$

This relation can be written as a polynomial of degree $n-1$ in σ_n,

$$\sum_{i=1,\pm}^{k}\sum_{q=k+1,-\pm}^{n-1}((u_ju_q))\langle jn\rangle[qn]\prod_{\substack{A=1,\pm\\A\neq j,q}}^{n-1}\sigma_{An}=0\,. \tag{7.41}$$

Thus, the number of solutions from this dominance balance pick up the term

$$(n-1)N^{(1)}_{\text{pol}}(n-1)\,. \tag{7.42}$$

Note that this term is independent of the MHV degree.

7.2.3 Balance 2: Bubbling Off

Now, we will proceed in a similar manner to the second balance of the proof presented in the last chapter. This dominance balance can be achieved by reparametrizing σ_n and $\sigma_\pm$, as done in (6.37) and (6.38). We can first consider the equations for i's and p's,

$$\begin{aligned}\mathcal{E}_{i\alpha}&=u_{ia}\Big(\sum_{j=1,j\neq i}^{k}\frac{u^a_j\kappa_{j\alpha}}{\sigma_{ji}}+\frac{u^a_+\kappa^1_{+\alpha}}{\sigma_{Ji}}\Big(1+\tilde\varepsilon\frac{X_+}{\sigma_{Ji}}\Big)+\frac{u^a_-\kappa^2_{-\alpha}}{\sigma_{Ji}}\Big(1+\tilde\varepsilon\frac{X_-}{\sigma_{Ji}}\Big)\Big)-v_{i2}\kappa_{i\alpha}\\&=u_{ia}\Big(\sum_{j=1,j\neq i}^{k}\frac{u^a_j\kappa_{j\alpha}}{\sigma_{ji}}+\frac{1}{\sigma_{Ji}}\big(u^a_+\kappa^1_{+\alpha}+u^a_-\kappa^2_{-\alpha}\big)+\mathcal{O}(\tilde\varepsilon)\Big)-v_{i2}\kappa_{i\alpha}\end{aligned} \tag{7.43}$$

$$\begin{aligned}\tilde{\mathcal{E}}_{i\dot\alpha}&=u_{ia}\Big(\sum_{q=k+1}^{n-1}-\frac{u^a_q\tilde\kappa_{q\alpha}}{\sigma_{qi}}-\tau\frac{u^a_n\tilde\kappa_{n\alpha}}{\sigma_{Ji}}\Big(1+\tilde\varepsilon\frac{X_n}{\sigma_{Ji}}\Big)+\frac{u^a_+\tilde\kappa^1_{+\alpha}}{\sigma_{Ji}}\Big(1+\tilde\varepsilon\frac{X_+}{\sigma_{Ji}}\Big)+\frac{u^a_-\tilde\kappa^2_{-\alpha}}{\sigma_{Ji}}\Big(1+-\tilde\varepsilon\frac{X_-}{\sigma_{Ji}}\Big)\Big)-\tilde\kappa_{i\alpha}\\&=u_{ia}\Big(\sum_{q=k+1}^{n-1}-\frac{u^a_q\tilde\kappa_{q\alpha}}{\sigma_{qi}}-\tau\frac{u^a_n\tilde\kappa_{n\alpha}}{\sigma_{Ji}}+\frac{1}{\sigma_{Ji}}\big(u^a_+\tilde\kappa^1_{+\alpha}+u^a_-\tilde\kappa^2_{-\alpha}\big)+\mathcal{O}(\tilde\varepsilon)+\mathcal{O}(\tilde{\tau\varepsilon})\Big)-\tilde\kappa_{i\alpha}\end{aligned} \tag{7.44}$$

$$\mathcal{E}_{p\alpha}=u_{pa}\Big(\sum_{j=1}^{k}\frac{u^a_j\kappa_{j\alpha}}{\sigma_{jp}}+\frac{1}{\sigma_{Jp}}\big(u^a_+\kappa^1_{+\alpha}+u^a_-\kappa^2_{-\alpha}\big)+\mathcal{O}(\tilde\varepsilon)\Big)+\kappa_{p\alpha} \tag{7.45}$$

$$\tilde{\mathcal{E}}_{p\dot\alpha}=u_{pa}\Big(\sum_{q=k+1,q\neq p}^{n-1}-\frac{u^a_q\tilde\kappa_{q\alpha}}{\sigma_{qp}}-\tau\frac{u^a_n\tilde\kappa_{n\alpha}}{\sigma_{Jp}}+\frac{1}{\sigma_{Ji}}\big(u^a_+\tilde\kappa^1_{+\alpha}+u^a_-\tilde\kappa^2_{-\alpha}\big)+\mathcal{O}(\tilde\varepsilon)+\mathcal{O}(\tilde{\tau\varepsilon})\Big)-v_{p1}\tilde\kappa_{p\alpha}\,, \tag{7.46}$$

and as we have expected, their n-dependent terms drop off under the limit and their $\pm$ terms group, similar to the loop scattering equations. The loop polarized scattering equations for particles $1,...,n-1$ reduced to the loop polarized scattering equations for

[6] The notation $-\pm$ in the sum means that the term with index $\pm$ must be added with a negative sign.

$n-1$ external particles.

Using the same procedure, we consider the loop polarized scattering equations for the n-th particle,

$$\begin{aligned}
\mathcal{E}_{n\alpha} &= u_{na}\Big(\sum_{j=1}^{k}\frac{u_j^a\kappa_{j\alpha}}{\sigma_{jJ}}\Big(1-\tilde{\varepsilon}\frac{X_n}{\sigma_{jJ}}\Big)+\frac{1}{\tilde{\tilde{\varepsilon}}}\Big(\frac{u_+^a\kappa_{+\alpha}^1}{X_{+n}}+\frac{u_-^a\kappa_{-\alpha}^2}{X_{-n}}\Big)\Big)+\tau\kappa_{n\alpha}\\
&= u_{na}\Big(\sum_{j=1}^{k}\frac{u_j^a\kappa_{j\alpha}}{\sigma_{jJ}}+\frac{1}{\tilde{\tilde{\varepsilon}}}\Big(\frac{u_+^a\kappa_{+\alpha}^1}{X_{+n}}+\frac{u_-^a\kappa_{-\alpha}^2}{X_{-n}}\Big)+\mathcal{O}(\tilde{\varepsilon})\Big)+\tau\kappa_{n\alpha} \qquad (7.47)\\
\tilde{\mathcal{E}}_{n\dot{\alpha}} &= u_{na}\Big(\sum_{q=k+1}^{n-1}-\frac{u_q^a\tilde{\kappa}_{q\alpha}}{\sigma_{qJ}}\Big(1-\tilde{\varepsilon}\frac{X_n}{\sigma_{jJ}}\Big)+\frac{1}{\tilde{\tilde{\varepsilon}}}\Big(\frac{u_+^a\tilde{\kappa}_{+\alpha}^1}{X_{+n}}+\frac{u_-^a\tilde{\kappa}_{-\alpha}^2}{X_{-n}}\Big)\Big)-\tau v_{n1}\tilde{\kappa}_{n\alpha}\\
&= u_{na}\Big(\sum_{q=k+1}^{n-1}-\frac{u_q^a\tilde{\kappa}_{q\alpha}}{\sigma_{qJ}}+\frac{1}{\tilde{\tilde{\varepsilon}}}\Big(\frac{u_+^a\tilde{\kappa}_{+\alpha}^1}{X_{+n}}+\frac{u_-^a\tilde{\kappa}_{-\alpha}^2}{X_{-n}}\Big)+\mathcal{O}(\tilde{\varepsilon})\Big)-\tau v_{n1}\tilde{\kappa}_{n\alpha}. \qquad (7.48)
\end{aligned}$$

In the loop scattering equations, the n^{th} particle's equations under the limit $\tau\to 0$ result in a polynomial of degree $(n-3)$ in σ_J. We expect that the end result of this to also be a degree $(n-3)$ polynomial in σ_J.

The loop polarized scattering equations for the off-shell particles under the reparametrization are given by

$$\begin{aligned}
\mathcal{E}_{\pm\alpha} &= u_{\pm,a}\Big(\sum_{j=1}^{k}\frac{u_j^a\kappa_{j\alpha}}{\sigma_{jJ}}\Big(1-\tilde{\varepsilon}\frac{X_\pm}{\sigma_{jJ}}\Big)+\frac{1}{\tilde{\tilde{\varepsilon}}}\frac{u_\mp^a\kappa_{\mp\alpha}^c}{X_{\mp\pm}}\Big)-v_{\pm,a}\kappa_{\pm\alpha}^a\\
&= u_{\pm,a}\Big(\sum_{j=1}^{k}\frac{u_j^a\kappa_{j\alpha}}{\sigma_{jJ}}+\frac{1}{\tilde{\tilde{\varepsilon}}}\frac{u_\mp^a\kappa_{\mp\alpha}^c}{X_{\mp\pm}}+\mathcal{O}(\tilde{\varepsilon})\Big)-v_{\pm,a}\kappa_{\pm\alpha}^a \qquad (7.49)\\
\tilde{\mathcal{E}}_{\pm\dot{\alpha}} &= u_{\pm,a}\Big(\sum_{q=k+1}^{n-1}-\frac{u_q^a\tilde{\kappa}_{q\alpha}}{\sigma_{qJ}}\Big(1-\tilde{\varepsilon}\frac{X_\pm}{\sigma_{qJ}}\Big)-\frac{\tau}{\tilde{\tilde{\varepsilon}}}\frac{u_n^a\tilde{\kappa}_{n\alpha}}{X_{n\pm}}+\frac{1}{\tilde{\tilde{\varepsilon}}}\frac{u_\mp^a\tilde{\kappa}_{\mp\alpha}^c}{X_{\mp\pm}}\Big)-v_{\pm,a}\tilde{\kappa}_{\pm\alpha}^a\\
&= u_{\pm,a}\Big(\sum_{q=k+1}^{n-1}-\frac{u_q^a\tilde{\kappa}_{q\alpha}}{\sigma_{qJ}}-\frac{\tau}{\tilde{\tilde{\varepsilon}}}\frac{u_n^a\tilde{\kappa}_{n\alpha}}{X_{n\pm}}+\frac{1}{\tilde{\tilde{\varepsilon}}}\frac{u_\mp^a\tilde{\kappa}_{\mp\alpha}^c}{X_{\mp\pm}}+\mathcal{O}(\tilde{\varepsilon})\Big)-v_{\pm,a}\tilde{\kappa}_{\pm\alpha}^a\,. \qquad (7.50)
\end{aligned}$$

For the loop scattering equations, the terms associated to the off-shell particles drop out in $i=1,...,n-1$ equations. For the loop polarized scattering equations, this can also be achieved by having $u_\pm\to s\hat{u}_\pm$ where s is a soft factor depending on τ and $\tilde{\varepsilon}$. Under this scaling, the polarized scattering equations become

$$\mathcal{E}_{i\alpha} = \sum_{j=1, j\neq i}^{k} \frac{((u_i u_j))\kappa_{j\alpha}}{\sigma_{ji}} + \frac{s}{\sigma_{Ji}}\big(((u_i \hat{u}_+))\kappa^1_{+\alpha} + ((u_i \hat{u}_-))\kappa^2_{-\alpha}\big) - v_{i2}\kappa_{i\alpha} \tag{7.51}$$

$$\tilde{\mathcal{E}}_{i\dot\alpha} = \sum_{q=k+1}^{n-1} -\frac{((u_i u_q))\tilde\kappa_{q\dot\alpha}}{\sigma_{qi}} - \tau\frac{((u_i u_n))\tilde\kappa_{n\dot\alpha}}{\sigma_{Ji}} + \frac{s}{\sigma_{Ji}}\big(((u_i \hat{u}_+))\tilde\kappa^1_{+\dot\alpha} + ((u_i \hat{u}_-))\tilde\kappa^2_{-\dot\alpha}\big) - \tilde\kappa_{i\dot\alpha} \tag{7.52}$$

$$\mathcal{E}_{n\alpha} = \sum_{j=1}^{k} \frac{((u_n u_j))\kappa_{j\alpha}}{\sigma_{jJ}} + \frac{s}{\tilde\varepsilon}\Big(\frac{((u_n \hat{u}_+))\kappa^1_{+\alpha}}{X_{+n}} + \frac{((u_n \hat{u}_-))\kappa^2_{-\alpha}}{X_{-n}}\Big) + \tau\kappa_{n\alpha} \tag{7.53}$$

$$\tilde{\mathcal{E}}_{n\dot\alpha} = \sum_{q=k+1}^{n-1} -\frac{((u_n u_q))\tilde\kappa_{q\dot\alpha}}{\sigma_{qJ}} + \frac{s}{\tilde\varepsilon}\Big(\frac{((u_n \hat{u}_+))\tilde\kappa^1_{+\dot\alpha}}{X_{+n}} + \frac{((u_n \hat{u}_-))\tilde\kappa^2_{-\dot\alpha}}{X_{-n}}\Big) - \tau v_{n1}\tilde\kappa_{n\dot\alpha} \tag{7.54}$$

$$\mathcal{E}_{\pm\alpha} = s\sum_{j=1}^{k} \frac{((\hat{u}_\pm u_j))\kappa_{j\alpha}}{\sigma_{jJ}} + \frac{s^2}{\tilde\varepsilon}\frac{((\hat{u}_\pm \hat{u}_\mp))\kappa^c_{\mp\alpha}}{X_{\mp\pm}} - v_{\pm,a}\kappa^a_{\pm\alpha} \tag{7.55}$$

$$\tilde{\mathcal{E}}_{\pm\dot\alpha} = -s\sum_{q=k+1}^{n-1} \frac{((\hat{u}_\pm u_q))\tilde\kappa_{q\dot\alpha}}{\sigma_{qJ}} - \frac{s\tau}{\tilde\varepsilon}\frac{((\hat{u}_\pm u_n))\tilde\kappa_{n\dot\alpha}}{X_{n\pm}} + \frac{s^2}{\tilde\varepsilon}\frac{((\hat{u}_\pm \hat{u}_\mp))\tilde\kappa^c_{\mp\dot\alpha}}{X_{\mp\pm}} - v_{\pm,a}\tilde\kappa^a_{\pm\dot\alpha}\,. \tag{7.56}$$

To determine how s depends on $\tilde\varepsilon$, we can look at the constraints that we have on them. There are two terms that we must be careful of in determining the weight of s, they are

$$\frac{s^2}{\tilde\varepsilon}\frac{((\hat{u}_\pm \hat{u}_\mp))}{X_\pm}\kappa^c_{\mp\alpha}\,, \tag{7.57}$$

from $\pm$ particle's equations and

$$\frac{s}{\tilde\varepsilon}\Big(\frac{((u_n \hat{u}_+))\tilde\kappa^1_{+\alpha}}{X_{+n}} + \frac{((u_n \hat{u}_-))\tilde\kappa^2_{-\alpha}}{X_{-n}}\Big) = 0\,, \tag{7.58}$$

from the n^{th} equations. We imposed that they must vanish at the leading order, similar to the situation in loop scattering equations. These equations diverge for $s < \tilde\varepsilon$. Even though the first equation will not diverge for $\tilde\varepsilon^{1/2} < s < \tilde\varepsilon$, the equation (7.58) is not satisfied since the terms inside the parenthesis cannot cancel with each other because the κ's are linearly independent. This suggest that s must be of order $\tilde\varepsilon$. For the homogeneity of the constraints $\mathcal{E}_n$ and $\tilde{\mathcal{E}}_n$, we must also have that $u_n \to \tau\hat{u}_n$.

In $\mathcal{E}_\pm$ and $\tilde{\mathcal{E}}_\pm$ with $s \sim \tilde\varepsilon$,

$$\mathcal{E}_{\pm\alpha} = \tilde\varepsilon\Big(\sum_{j=1}^{k} \frac{((\hat{u}_\pm u_j))\kappa_{j\alpha}}{\sigma_{jJ}}\Big) + \tilde\varepsilon\frac{((\hat{u}_\pm \hat{u}_\mp))\kappa^c_{\mp\alpha}}{X_{\pm\mp}} - v_{\pm a}\kappa^a_{\pm\alpha} \tag{7.59}$$

$$\tilde{\mathcal{E}}_{\pm\dot\alpha} = -\tilde\varepsilon\Big(\sum_{q=k+1}^{n-1} \frac{((\hat{u}_\pm u_q))\tilde\kappa_{q\dot\alpha}}{\sigma_{qJ}}\Big) - \tau^2\frac{((\hat{u}_\pm \hat{u}_n))\tilde\kappa_{n\dot\alpha}}{X_{n\pm}} + \tilde\varepsilon\frac{((\hat{u}_\pm \hat{u}_\mp))\tilde\kappa^c_{\mp\dot\alpha}}{X_{\mp\pm}} - v_{\pm a}\tilde\kappa^a_{\pm\dot\alpha}. \tag{7.60}$$

one can see that every terms, except the last, contain soft factors. Under the limit $\tilde\varepsilon \to 0$ and $\tau \to 0$, all terms would vanish and this would force $v_\pm = 0$. We must then find a dominance balance that does not let every term except the last vanish i.e. one of the term become of order $\mathcal{O}(1)$. The dominance balance can be achieved by reparametrizing for the second time

$$X_+ = X_+ + \hat{\varepsilon} Y_+ \tag{7.61}$$
$$X_- = X_+ + \hat{\varepsilon} Y_- \tag{7.62}$$

with $Y_+ = 0$ as we have done in the previous section. $\mathcal{E}_+$ then have two terms that could possibly be of $\mathcal{O}(1)$:

$$\frac{\tilde{\varepsilon}}{\hat{\varepsilon}} \frac{((\hat{u}_+ \hat{u}_-))\kappa^2_{-\alpha}}{Y_-} - v_{+a}\kappa^a_{+\alpha} = 0\,, \tag{7.63}$$

which can be achieved by requiring $\tilde{\varepsilon}/\hat{\varepsilon} \sim 1$. By independence of the kappas, one must have that

$$0 = -\frac{((\hat{u}_+ \hat{u}_-))}{Y_-} - v_{+2} \quad \text{and} \quad 0 = v_{+1}\,. \tag{7.64}$$

With the normalization condition $((v_+ \epsilon_+)) = 1$, or explicitly $v_{+a} = (v_{+1}, -1)$, the first equation becomes

$$\frac{((\hat{u}_+ \hat{u}_-))}{Y_-} = 1\,, \tag{7.65}$$

giving a nice constraint relating the products of $u_\pm$ and the marked point Y_-. We can proceed in a similar fashion to $\mathcal{E}_-$, giving the conditions

$$0 = \frac{((\hat{u}_- \hat{u}_+))}{-Y_-} - v_{-1} \quad \text{and} \quad 0 = v_{-2}\,. \tag{7.66}$$

With the normalization condition $((v_- \epsilon_-)) = 1$, or explicitly $v_{-a} = (1, v_{-2})$, this give the same relation

$$\frac{((\hat{u}_+ \hat{u}_-))}{Y_-} = 1\,. \tag{7.67}$$

Now that we have done the second reparametrization, we want to find some relationship of the soft parameters $\tilde{\varepsilon}$ and τ. In the last chapter, we determined the order of the soft parameters by requiring the some terms in $\mathcal{E}_-$ to vanish. These are the terms that do not agree back-to-back with $\mathcal{E}_+$. In this case, we can explicitly substitute the second reparametrization and restore the $\mathcal{O}(\tilde{\varepsilon})$ in $\tilde{\mathcal{E}}_-$, this gives the relation

$$\begin{aligned} \tilde{\mathcal{E}}_{-\dot{\alpha}} = &-\tilde{\varepsilon} \sum_{q=k+1}^{n-1} \frac{((\hat{u}_- u_q))\tilde{\kappa}_{q\dot{\alpha}}}{\sigma_{qJ}} + \tilde{\varepsilon}^2 \sum_{q=k+1}^{n-1} \frac{X_+((\hat{u}_- u_q))\tilde{\kappa}_{q\dot{\alpha}}}{\sigma^2_{qJ}} - \tilde{\varepsilon}^3 \sum_{q=k+1}^{n-1} \frac{Y_-((\hat{u}_- u_q))\tilde{\kappa}_{q\dot{\alpha}}}{\sigma^2_{qJ}} \\ &- \tau^2 \frac{((\hat{u}_- \hat{u}_n))\tilde{\kappa}_{n\dot{\alpha}}}{X_{n+}} - \tau^2 \tilde{\varepsilon} \frac{Y_-((\hat{u}_- \hat{u}_n))\tilde{\kappa}_{n\dot{\alpha}}}{X^2_{n+}} + \frac{((\hat{u}_+ \hat{u}_-))\tilde{\kappa}^1_{+\dot{\alpha}}}{Y_-} - v_{-a}\tilde{\kappa}^a_{-\dot{\alpha}}\,. \end{aligned} \tag{7.68}$$

We can set aside the last two terms vanishes at $\mathcal{O}(1)$, the rest of the terms can be compared to those of $\tilde{\mathcal{E}}_+$. The terms that are not presented in $\tilde{\mathcal{E}}_+$ are the terms containing Y_- in the numerator. Requiring that they must vanish together at the same order imposes $\tilde{\varepsilon}^3 \sim \tau^2 \tilde{\epsilon}$, which simplifies to $\tilde{\varepsilon} \sim \tau$.[7]

[7] Note that τ here is actually $\tau^{1/2}$ of the τ in the loop scattering equations, since $k_n = \kappa_n \tilde{\kappa}_n$.

Now that we have obtained some relations from the constraints $\mathcal{E}_{\pm}$, we can proceed to find the solutions for σ_n, $u_{n,a}$, and v_{n1}. First, consider the n^{th} particle's polarized scattering equations, which are now

$$\mathcal{E}_{n\alpha}/\tau = \sum_{j=1}^{k} \frac{(\!(\hat{u}_n u_j)\!)\kappa_{j\alpha}}{\sigma_{jJ}} + \frac{(\!(\hat{u}_n \hat{u}_+)\!)\kappa^1_{+\alpha}}{X_{+n}} + \frac{(\!(\hat{u}_n \hat{u}_-)\!)\kappa^2_{-\alpha}}{X_{+n}} - \tilde{\varepsilon}\frac{Y_-(\!(\hat{u}_n \hat{u}_-)\!)\kappa^2_{-\alpha}}{X^3_{+n}} + \kappa_{n\alpha} \tag{7.69}$$

$$\tilde{\mathcal{E}}_{n\dot{\alpha}}/\tau = \sum_{q=k+1}^{n-1} -\frac{(\!(\hat{u}_n u_q)\!)\tilde{\kappa}_{q\dot{\alpha}}}{\sigma_{qJ}} + \frac{\Big((\!(\hat{u}_n \hat{u}_+)\!)\tilde{\kappa}^1_{+\dot{\alpha}}}{X_{+n}} + \frac{(\!(\hat{u}_n \hat{u}_-)\!)\tilde{\kappa}^2_{-\dot{\alpha}}}{X_{+n}} - \tilde{\varepsilon}\frac{Y_-(\!(\hat{u}_n \hat{u}_-)\!)\tilde{\kappa}^2_{-\dot{\alpha}}}{X^3_{+n}}\Big) - v_{n1}\tilde{\kappa}_{n\dot{\alpha}} \,. \tag{7.70}$$

These are four equations that can be used to solve for the four variables. Similar to the process we did with the refined scattering equations, we separate four of them by dotting in κ_n and κ_1 to $\mathcal{E}_n$, and $\tilde{\kappa}_n$ $\tilde{\kappa}_1$ to $\tilde{\mathcal{E}}_n$:

$$\mathcal{E}_{n\alpha}\kappa^{\alpha}_n/\tau = \sum_{j=1}^{k} \frac{(\!(\hat{u}_n u_j)\!)\langle jn\rangle}{\sigma_{jJ}} + \frac{1}{X_{+n}}\Big((\!(\hat{u}_n \hat{u}_+)\!)\langle +^1 n\rangle + (\!(\hat{u}_n \hat{u}_-)\!)\langle -^2 n\rangle\Big) \tag{7.71}$$

$$\mathcal{E}_{n\alpha}\kappa^{\alpha}_1/\tau = \sum_{j=2}^{k} \frac{(\!(\hat{u}_n u_j)\!)\langle j1\rangle}{\sigma_{jJ}} + \frac{1}{X_{+n}}\Big((\!(\hat{u}_n \hat{u}_+)\!)\langle +^1 1\rangle + (\!(\hat{u}_n \hat{u}_-)\!)\langle -^2 1\rangle\Big) + \langle n1\rangle \tag{7.72}$$

$$\tilde{\mathcal{E}}_{n\dot{\alpha}}\tilde{\kappa}^{\dot{\alpha}}_n/\tau = \sum_{q=k+1}^{n-1} -\frac{(\!(\hat{u}_n u_q)\!)\,[qn]}{\sigma_{qJ}} + \frac{1}{X_{+n}}\Big((\!(\hat{u}_n \hat{u}_+)\!)[+^1 n] + (\!(\hat{u}_n \hat{u}_-)\!)[-^2 n]\Big) \tag{7.73}$$

$$\tilde{\mathcal{E}}_{n\dot{\alpha}}\tilde{\kappa}^{\dot{\alpha}}_1/\tau = \sum_{q=k+1}^{n-1} -\frac{(\!(\hat{u}_n u_q)\!)\,[q1]}{\sigma_{qJ}} + \frac{1}{X_{+n}}\Big((\!(\hat{u}_n \hat{u}_+)\!)[+^1 1] + (\!(\hat{u}_n \hat{u}_-)\!)[-^2 1]\Big) - v_{n1}\,[n1] \,. \tag{7.74}$$

The last equation, (7.74), can then be used to solve for v_{n1}:

$$v_{n1} = [n1]\left(\sum_{q=k+1}^{n-1} -\frac{((\hat{u}_n u_q))\,[q1]}{\sigma_{qJ}} + \frac{1}{X_{+n}}\Big(((\hat{u}_n \hat{u}_+))[+^1 1] + ((\hat{u}_n \hat{u}_-))[-^2 1]\Big)\right). \tag{7.75}$$

Like in the previous section, we can use either (7.71) or (7.73) to solve for a relation between $\hat{u}^1_n$ and $\hat{u}^2_n$. Here, (7.71) were chosen to be used :

$$0 = \sum_{j=1}^{k} \frac{\langle jn\rangle}{\sigma_{jJ}}(\hat{u}^1_n u^2_j - \hat{u}^2_n u^1_j) + \frac{1}{X_{+n}}\Big((\hat{u}^1_n \hat{u}^2_+ - \hat{u}^2_n \hat{u}^1_+)\langle +^1 n\rangle + (\hat{u}^1_n \hat{u}^2_- - \hat{u}^2_n \hat{u}^1_-)\langle -^2 n\rangle\Big)\,, \tag{7.76}$$

giving us the relation

$$\hat{u}^1_n = \hat{u}^2_n \frac{\sum_{j=1}^{k} \frac{\langle jn\rangle}{\sigma_{jJ}} u^1_j + \frac{1}{X_{+n}}\Big(\hat{u}^1_+\,\langle +^1 n\rangle + \hat{u}^1_-\,\langle -^2 n\rangle\Big)}{\sum_{j=1}^{k} \frac{\langle jn\rangle}{\sigma_{jJ}} u^2_j + \frac{1}{X_{+n}}\Big(\hat{u}^2_+\,\langle +^1 n\rangle + \hat{u}^2_-\,\langle -^2 n\rangle\Big)}. \tag{7.77}$$

Substituting this relation into (7.73) and with some calculation, we obtain

$$
\begin{aligned}
0 = &\sum_{j=1}^{k}\sum_{q=k+1}^{n-1}\frac{[nq]}{\sigma_{qJ}}\frac{\langle jn\rangle}{\sigma_{jJ}}((u_j u_q)) + \frac{1}{X_{+n}}\sum_{q=k+1}^{n-1}\frac{[nq]}{\sigma_{qJ}}\Big(((\hat{u}_+ u_q))\langle +^1 n\rangle + ((\hat{u}_- u_q))\langle -^2 n\rangle\Big) \\
&+ \frac{1}{X_{+n}}\sum_{j=1}^{k}\frac{\langle jn\rangle}{\sigma_{jJ}}\Big(((u_j\hat{u}_+))[+^1 n] + ((u_j\hat{u}_-))[-^2 n]\Big) \\
&+ \frac{1}{X_{+n}^2}\Big(((\hat{u}_-\hat{u}_+))\langle -^2 n\rangle[+^1 n] + ((\hat{u}_+\hat{u}_-))\langle +^1 n\rangle[-^2 n]\Big)\,. \qquad (7.78)
\end{aligned}
$$

This equation can be tremendously simplified using the other loop scattering equations. First, consider the term with $1/X_{+n}$:

$$
\frac{1}{X_{+n}}\left[\sum_{q=k+1}^{n-1}\frac{[nq]}{\sigma_{qJ}}\Big(((\hat{u}_+ u_q))\langle +^1 n\rangle + ((\hat{u}_- u_q)v)\langle -^2 n\rangle\Big) + \sum_{j=1}^{k}\frac{\langle jn\rangle}{\sigma_{jJ}}\Big(((u_j\hat{u}_+))[+^1 n] + ((u_j\hat{u}_-))[-^2 n]\Big)\right],
$$

The left term looks like a part of $[n\tilde{\mathcal{E}}_+]$:

$$
[n\tilde{\mathcal{E}}_+] = \sum_{q=k+1}^{n-1} -\frac{((u_+u_q))[nq]}{\sigma_{q+}} + \frac{((u_+u_-))[n -^2]}{\sigma_{-+}} - v_{+1}[n +^1] - v_{+2}[n +^2]\,,
$$

using the new parametrizations: $\sigma_\pm = \sigma_J + \tau X_+ + \tau^2 Y_\pm$ where $Y_+ = 0$, this equation becomes

$$
[n\tilde{\mathcal{E}}_+] = \sum_{q=k+1}^{n-1} -\frac{((u_+u_q))[nq]}{\sigma_{qJ}} + \frac{((u_+u_-))[n -^2]}{\tau^2 Y_-} - v_{+1}[n +^1] - v_{+2}[n +^2]\,.
$$

Now, we use the scaling and form of the variables obtained previously, which are $u_\pm \sim \tau\hat{u}_\pm$, $v_{+1} = 0 + \tau\hat{v}_{+1}$, and $v_{+2} = -((\hat{u}_+\hat{u}_-))/Y_- = -1$, this allow us to group some terms together as

$$
-\tau\Big(\sum_{q=k+1}^{n-1} -\frac{((\hat{u}_+u_q))[nq]}{\sigma_{qJ}} + \hat{v}_{+1}[n +^1]\Big) + \frac{((\hat{u}_+\hat{u}_-))[n -^2]}{Y_-} + \frac{((\hat{u}_+\hat{u}_-))}{Y_-}[n +^2]. \qquad (7.79)
$$

The two terms on the right of order $\mathcal{O}(1)$ cancel each other, and what's left from the $\mathcal{O}(\tau)$ gives

$$
\sum_{q=k+1}^{n-1} -\frac{((\hat{u}_+u_q))[nq]}{\sigma_{qJ}} = -\hat{v}_{+1}[n +^1]. \qquad (7.80)
$$

We can repeat the same procedure on $[n\tilde{\mathcal{E}}_-]$, $\langle\mathcal{E}_+ n\rangle$, and $\langle\mathcal{E}_- n\rangle$, giving the relations

$$
\begin{aligned}
\sum_{q=k+1}^{n-1}\frac{((\hat{u}_-u_q))\,[nq]}{\sigma_{qJ}} &= \hat{v}_{-2}\,[n+^2] \\
\sum_{j=1}^{k}\frac{((\hat{u}_+u_j))\,\langle jn\rangle}{\sigma_{jJ}} &= \hat{v}_{+1}\,\langle +^1 n\rangle \\
\sum_{j=1}^{k}\frac{((\hat{u}_-u_j))\,\langle jn\rangle}{\sigma_{jJ}} &= -\hat{v}_{-2}\,\langle +^2 n\rangle\,. \qquad (7.81)
\end{aligned}
$$

Substituting all of these into the original equation, one can see that this term vanishes:

$$-\hat{v}_{+1}\left[n+^1\right]\left\langle +^1 n\right\rangle+\hat{v}_{-2}\left[n+^2\right]\left\langle -^2 n\right\rangle-\hat{v}_{+1}\left\langle +^1 n\right\rangle\left[+^1 n\right]+\hat{v}_{-2}\left\langle +^2 n\right\rangle\left[-^2 n\right]=0\,. \tag{7.82}$$

Therefore, the second and the third sum in (7.78) cancels each other.

For the last term in (7.78) with X_{+n}^{-2}, we can rewrite them without the factor $((\hat{u}_-\hat{u}_+))$ as

$$\begin{aligned}\left\langle -^2 n\right\rangle\left[+^1 n\right]-\left\langle +^1 n\right\rangle\left[-^2 n\right]&=-\left\langle +^2 n\right\rangle\left[+^1 n\right]+\left\langle +^1 n\right\rangle\left[+^2 n\right]\\&=\left\langle +^1 n\right\rangle\left[n+^2\right]-\left\langle +^2 n\right\rangle\left[n+^1\right]\\&=k_{n\alpha\dot{\alpha}}\left(\kappa_+^{1\alpha}\tilde{\kappa}_+^{2\dot{\alpha}}-\kappa_+^{2\alpha}\tilde{\kappa}_+^{1\dot{\alpha}}\right)\\&=k_{n\alpha\dot{\alpha}}\epsilon_{ab}\kappa_+^{a\alpha}\tilde{\kappa}_+^{b\dot{\alpha}}=k_n\cdot l\,.\end{aligned} \tag{7.83}$$

The equation in (7.78) can now be simplified to

$$\sum_{j=1}^{k}\sum_{q=k+1}^{n-1}\frac{[nq]}{\sigma_{qJ}}\frac{\langle jn\rangle}{\sigma_{jJ}}((u_ju_q))+\frac{((\hat{u}_+\hat{u}_-))}{X_{+n}^2}k_n\cdot l\,, \tag{7.84}$$

and as a polynomial in σ_n, with $((\hat{u}_+\hat{u}_-))=Y_-$,

$$\sum_{j=1}^{k}\sum_{q=k+1}^{n-1}\langle jn\rangle[nq]((u_ju_q))\prod_{\substack{A=1\\A\neq j,q}}^{n-1}\sigma_{AJ}+\frac{Y_-}{X_{+n}^2}k_n\cdot l\prod_{A=1}^{n-1}\sigma_{AJ}\,. \tag{7.85}$$

The residual term looks exactly like in bubbling off the CHY scattering equations in (6.41). Currently, (7.85) is not the expected polynomial of degree $(n-3)$ because of the residual term, which we expected it to cancel with the solution of the other dominance balance.

For completeness, we can solve for $\hat{u}_n^2$ by substituting the relation (7.77) into (7.72). After some calculation, we have

$$\begin{aligned}0=\hat{u}_n^2\Bigg[&\sum_{i=1}^{k}\sum_{j=2}^{k}\frac{\langle j1\rangle\langle in\rangle}{\sigma_{jJ}\sigma_{iJ}}((u_iu_j))+\frac{1}{X_{+n}}\left(\sum_{j=2}^{k}\frac{((\hat{u}_+u_j))}{\sigma_{jJ}}\langle j1\rangle\left\langle +^1 n\right\rangle\right.\\&\left.+\sum_{j=2}^{k}\frac{((\hat{u}_-^1u_j))}{\sigma_{jJ}}\langle j1\rangle\left\langle -^2 n\right\rangle+\sum_{i=1}^{k}\frac{((u_i\hat{u}_+))}{\sigma_{iJ}}\left\langle +^1 1\right\rangle\langle in\rangle+\sum_{i=1}^{k}\frac{((u_i\hat{u}_-))}{\sigma_{iJ}}\left\langle -^2 1\right\rangle\langle in\rangle\right)\\&+\frac{1}{X_{+n}^2}\left(((\hat{u}_-\hat{u}_+))\left\langle +^1 1\right\rangle\left\langle -^2 n\right\rangle+((\hat{u}_+\hat{u}_-))\left\langle +^1 n\right\rangle\left\langle -^2 1\right\rangle\right)\Bigg]\\&+\left(\sum_{j=1}^{k}\frac{\langle jn\rangle}{\sigma_{jJ}}u_i^2+\frac{1}{X_{+n}}\left(\hat{u}_+^2\left\langle +^1 n\right\rangle+\hat{u}_-^2\left\langle -^2 n\right\rangle\right)\right)\langle n1\rangle\,.\end{aligned} \tag{7.86}$$

Similar to the previous derivation, the terms with X_{+n}^{-1} vanish via the relations (7.80) and (7.81). For the term with $1/X_{+n}^2$, one can use the Schouten identity to simplify the terms:

$$\begin{aligned}
&((\hat{u}_-\hat{u}_+))\big\langle +^1\, 1\big\rangle\big\langle -^2\, n\big\rangle + ((\hat{u}_+\hat{u}_-))\big\langle +^1\, n\big\rangle\big\langle -^2\, 1\big\rangle \\
&\quad = ((\hat{u}_+\hat{u}_-))\Big(\big\langle +^1\, 1\big\rangle\big\langle +^2\, n\big\rangle + \big\langle +^1\, n\big\rangle\big\langle 1\, +^2\,\big\rangle\Big) \\
&\quad = ((\hat{u}_+\hat{u}_-))\big\langle +^1\, +^2\big\rangle\big\langle n1\big\rangle\,.
\end{aligned} \tag{7.87}$$

Thus, (7.86) simplifies and give a solution for $\hat{u}_n^2$

$$\hat{u}_n^2 = \big\langle 1n\big\rangle \frac{\sum_{j=1}^k \frac{\langle jn\rangle}{\sigma_{jJ}} u_i^2 + \frac{1}{X_{+n}}\Big(\hat{u}_+^2\,\langle +^1 n\rangle + \hat{u}_-^2\,\langle -^2 n\rangle\Big)}{\sum_{i=1}^k\sum_{j=2}^k \frac{\langle j1\rangle\langle in\rangle}{\sigma_{jJ}\sigma_{iJ}}((u_i u_j)) + \frac{1}{X_{+n}^2}((\hat{u}_+\hat{u}_-))\,\langle +^1 +^2\rangle\,\langle n1\rangle}\,, \tag{7.88}$$

and $\hat{u}_n^1$ can be obtained by the relation (7.77)

$$\hat{u}_n^1 = \big\langle 1n\big\rangle \frac{\sum_{j=1}^k \frac{\langle jn\rangle}{\sigma_{jJ}} u_i^1 + \frac{1}{X_{+n}}\Big(\hat{u}_+^1\,\langle +^1 n\rangle + \hat{u}_-^1\,\langle -^2 n\rangle\Big)}{\sum_{i=1}^k\sum_{j=2}^k \frac{\langle j1\rangle\langle in\rangle}{\sigma_{jJ}\sigma_{iJ}}((u_i u_j)) + \frac{1}{X_{+n}^2}((\hat{u}_+\hat{u}_-))\,\langle +^1 +^2\rangle\,\langle n1\rangle}\,. \tag{7.89}$$

The calculation that has been completed up to this point has found one of the term in the recursion relation for the number of solutions from the first dominance balance. The term is of the form $(n-1)\mathcal{N}^{(1)}(n-1)$, resembling the term obtained in the first dominance balance in the last chapter. Up to this point, the polynomial in σ_n solved in second dominance balance have not give the term that we have been expecting. In fact, the residual term with $k_n \cdot l$ resembles the term that vanishes under the soft limit of the loop scattering equations.

Chapter 8

Conclusion and Discussion

In this project, we presented the proof for the conjecture on the number of solutions of the refined scattering equations that it is the Eulerian numbers. Moreover, we proved in details for the number of solutions to the loop scattering equation using two methods. Lastly, we showed some work in progress toward the proof for the one-loop polarized scattering equations.

Preceding the proofs, this project reviews the physical and mathematical foundations of scattering amplitude and twistor theory. These include the spinor-helicity formalism that provides understanding on the MHV sectors of the amplitude. Then, the twistor theory was reviewed to lay the foundation for describing general helicity amplitudes using RSVW formula.

The number of solutions to the refined scattering equation is known in the field to be the Eulerian number $E(n-3,k-2)$. However, there was no explicit proof of the conjecture. The first objective of this project is to show that the number of solutions of refined scattering equation satisfies the recursion relation for the Eulerian numbers. This was proved by induction, using the method of dominance balance to solve for the number of solutions in each sector. In chapter 5, it was shown that there are two dominance balances of the refined scattering equations, and the results from solving the two balances are degree $n-k-1$ and $k-1$ polynomial in σ_n. The number of solutions of σ_n gives the recursion relation $\mathcal{N}^{(0)}(n,k) = (k-1)\mathcal{N}^{(0)}(n-1,k) + (n-k-1)\mathcal{N}^{(0)}(n-1,k-1)$, which is the same recursion relation for the Eulerian numbers.

After proving the tree-level number of solution, we ambitiously head on to use a similar method at one loop. The one loop amplitude in 4d can be obtained from a tree level amplitude in high dimension by dimensional reduction. We choose to work with two formulations of the higher dimensional amplitude, that is using the CHY (chapter 6) and the 6d twistor formulation (chapter 7).

Chapter 6 provides a short introduction to the CHY formula, in which the scattering equations play an important role. The scattering equations, like the refined scattering equations, capture the localization features on the Riemann sphere, but they depend on the momentum rather than the spinor decomposition of the momentum. This allows the CHY formalism to work for arbitrary dimensions. The one-loop extension of the

CHY can be achieved by working on a nodal Riemann sphere, where the nodal points add two more marked points to the sphere. The momentum was embedded in higher dimensions, so that the 4d massive loop momentum is “on-shell”, and the higher dimension scattering equations can be used. Embedding the high dimension scattering equations into 4d loop scattering equations requires the forward limit so that the loop momentum flows in and out of the two nodal points. We take the soft limit of the n^{th} particle and the forward limit, then solve for the expression for the variables associated to the n^{th} particle for each dominance balance. This allow us to establish a recursion relation for the one loop scattering equations by counting the number of solutions for each dominance balance. The number of solutions for the one-loop CHY constraints was obtained as $\mathcal{N}^{(1)}(n) = (n-1)! - (n-2)! = (n-1)\mathcal{N}^{(1)}(n-1) + (n-3)\mathcal{N}^{(0)}(n-1)$, depending on the number of solutions to the $n-1$ particle amplitude of both one loop and tree level.

The 7^{th} chapter presented some work in progress on finding the number of solutions to the constraints of the one loop amplitude formula. The formula for one-loop super-amplitudes was recently published during the period where the project is ongoing, so it captures our interest to use similar techniques to establish a recursion relation for number of solutions. In this formalism, the one-loop amplitude was calculated using the off-shell tree amplitude in higher dimension, then reduce to 4d one-loop amplitude under a forward limit. The constraints of this formula are called the loop polarized scattering equations.

Up to this point, we have found a term in the recursion relations for the number of solutions from one of the dominance balances. It is of the form $(n-1)\mathcal{N}^{(1)}(n-1)$, resembling what we had from one of the dominance balance in the previous chapter. Although the number of solutions for the other dominance balance is still unclear, this dominance balance is achieved by reparametrizing the points σ_n and $\sigma_\pm$, also similar to what have been done in the previous chapter. These features show that the number of solutions has similar features to the one obtained for the loop CHY.

Also, the numerical results of four particle scattering give equal number of solutions for all MHV sectors. This is not clear if this is the case, since from supersymmetry, all plus/minus and one plus/minus amplitudes vanishes at all perturbative level. These results spark curiosity on the behaviour of the loop polarized scattering equations. We expected that the vanishing of these amplitudes must come from the integrand of the formula. Up to this point, the terms in the recursion relation do not appear to split by the MHV degree. Motivated by the refined scattering equations, we also expect the number of solutions to split by the MHV degree. This can be further investigated by considering the other dominance balances, such as switching the k and n particle.

Appendix A

Method of Dominance Balance

This appendix will present the method of dominance balance following [27], as it will be used in chapter 5-7.

There are two kinds of perturbation problems. The first kind is the one that most physics students learned quite early, that is, the regular perturbation problem, where the perturbation ε does not qualitative change the problem much. For some algebraic equations $P(x) = 0$, they are of the form $P(x) + \varepsilon = 0$, and the solution can be found by using a power series expansion of ε as an ansatz. The second kind is where the perturbation qualitatively change the problem. These can be solved asymptotically for a solution at different regions of ε called as the dominance balances. The expansion of the solution will depends singularly on ε.

A nice example given in [27] to distinguish these two problem is to consider an algebraic equation

$$x^3 - x = 0\,, \tag{A.1}$$

where the unperturbed solutions are $x = \pm 1, 0$. The regular perturbation of this equation is given by

$$x^3 - x + \varepsilon = 0\,, \tag{A.2}$$

and this can be solved by using a power series expansion

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots\,. \tag{A.3}$$

Plugging this into (A.2), matching the xs, we have the relations

$$\begin{aligned} \mathcal{O}(\varepsilon^0) &: x_0^3 - x_0 = 0 \\ \mathcal{O}(\varepsilon^1) &: 3x_0^2 x_1 - x_1 + 1 = 0 \\ \mathcal{O}(\varepsilon^2) &: 3x_0 x_2 - x_2 + 3x_0 x_1^2 = 0\,. \end{aligned} \tag{A.4}$$

(A.5)

Solving each of these would give

$$x_0 = -1, 0, 1 \tag{A.6}$$

$$x_1 = \frac{1}{1-3x_0^2} = \frac{1}{2}, 1, \frac{1}{2} \tag{A.7}$$

$$x_2 = \frac{3x_0x_1^2}{1-3x_0} = -\frac{3}{16}, 0, \frac{3}{8}\,, \tag{A.8}$$

$$\tag{A.9}$$

and so on, to any order that we want. Therefore, the three roots of the equation are

$$x = 0 + \varepsilon + \mathcal{O}(\varepsilon^3) \tag{A.10}$$

$$x = 1 + \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 + \mathcal{O}(\varepsilon^3) \tag{A.11}$$

$$x = -1 + \frac{1}{2}\varepsilon - \frac{3}{16}\varepsilon^2 + \mathcal{O}(\varepsilon^3)\,. \tag{A.12}$$

The regular perturbation of this equation can be done by, for example,

$$\varepsilon x^3 - x + 1 = 0\,. \tag{A.13}$$

Using the power series, one can get

$$\begin{aligned} \mathcal{O}(\varepsilon^0) &: x_0 - 1 = 0 \\ \mathcal{O}(\varepsilon^1) &: x_0^3 - x_1 = 0 \\ \mathcal{O}(\varepsilon^2) &: 3x_0^2x_1 - x_2 = 0\,, \end{aligned} \tag{A.14}$$

which gives only one of the three solution:

$$x = 1 + \varepsilon + 3\varepsilon^2 + \mathcal{O}(\varepsilon^3)\,. \tag{A.15}$$

The other two solutions are "hidden" in the limit ε and can be obtained by scaling the variable differently with powers of ε. We introduce a new rescaled variable y

$$x = \frac{y}{\delta(\varepsilon)} \tag{A.16}$$

and we want $y \sim \mathcal{O}(1)$ as $\varepsilon \to 0$. This delta will be the scaling factor that depends on some power of ε. Substituting (A.16) into (A.13), we now have

$$\frac{\varepsilon y^3}{\delta^3} - \frac{y}{\delta} + 1 = 0\,. \tag{A.17}$$

To get the non-trivial solutions, we must balance the terms, meaning that we will look the equation where some of its term are of the same order of magnitude. These balancing of the terms are call finding the dominance balance. The first dominance balance that will be chosen is to balance the first two terms, this mean that we want

$$\frac{\varepsilon}{\delta^3} = \frac{1}{\delta}\,, \tag{A.18}$$

giving $\delta \sim \varepsilon^{1/2}$. For this dominance balance, (A.17) can be written as

$$\frac{y^3}{\varepsilon^{1/2}} - \frac{y}{\varepsilon^{1/2}} + 1 = 0 \to y^3 - y + \varepsilon^{1/2} = 0\,. \tag{A.19}$$

Solving this using the power series of y, we get

$$y = \pm 1 - \frac{1}{2}\varepsilon^{1/2} + \mathcal{O}(\varepsilon)\,, \tag{A.20}$$

and for x

$$x = \pm\frac{1}{\varepsilon^{1/2}} - \frac{1}{2} + \mathcal{O}(\varepsilon^{1/2})\,. \tag{A.21}$$

These two solutions diverges at $\varepsilon \to 0$, so these two solutions cannot be obtained from solving directly as in the trivial case.

The other dominance balance that we can choose is to have the last two term be of the same order

$$\frac{1}{\delta} = 1\,, \tag{A.22}$$

giving $\delta \sim 1$. The solution for this case is just the one found in (A.15).

Choosing the first and the last to be of the same order gives

$$\frac{\varepsilon}{\delta^3} = 1\,, \tag{A.23}$$

so in this dominance balance $\delta \sim \varepsilon^{1/3}$, giving

$$y^3 - \frac{y}{\varepsilon^{1/3}} + 1 = 0\,, \tag{A.24}$$

with the second term of $\mathcal{O}(\varepsilon^{1/3})$, bigger than the terms that we balanced. This mean we do not obtained a dominance balance any new solutions. Note that in some problem, balancing all the terms is also a choice. Though, for this problem, it does not give a dominance balance.

Appendix B

Proof of Lemma 1

This appendix will present the proof to lemma 1, which states the following:

Lemma 1 The measure $d\mu_{n,k}$ is invariant of the choice of negative helicity particle and depend on just the number of negative helicity particle.

Proof. First of all, we will make the assumption that σ_i and κ_i are not transformed under the switching of the particle, since they should satisfied the same scattering equation

$$\sum_{j\neq i} \frac{k_i \cdot k_j}{\sigma_{ij}} = 0\,, \tag{B.1}$$

under the transformation.
We want to switch two particles' helicity to show that the measure is invariant under the relabelling of the particle. For this to coincide with the calculation in chapter 5, we switch particles k and n, so that now particle $n, 1, ..., k-1$ have negative helicity and $k, ..., n-1$ have positive helicity. Let υ_i be the variable u_i under the switch of the particle. Now, to see how this transformation map u_i to υ_i, we can find the relation between them. First, we can compare (5.4) and (5.15) by substituting the expression for κ_k from (5.5) into (5.4). By doing this, (5.4) became

$$\mathcal{E}_p = \sum_{i=1}^{k} \left(\frac{u_p u_i}{\sigma_{pi}} - \frac{u_p u_i \sigma_{nk}}{\sigma_{ni}\sigma_{pk}}\right)\kappa_i - \kappa_p + \frac{u_p}{u_n}\frac{\sigma_{nk}}{\sigma_{pk}}\kappa_n\,. \tag{B.2}$$

Comparing the coefficient of κ_n in (B.2) and (5.15) we obtain the relation

$$\frac{u_p}{u_n}\frac{\sigma_{nk}}{\sigma_{pk}} = \frac{\upsilon_p \upsilon_n}{\sigma_{pn}}\,. \tag{B.3}$$

We will see that υ_p must be a function of the variables u_p, σ_p, σ_n, and σ_k , otherwise it would not give a solution, which we will show later. Under the claim that $\upsilon_p = \upsilon_p(u_p, \sigma_p, \sigma_n, \sigma_k)$ and $\upsilon_n = \upsilon_n(u_n, \sigma_n, \sigma_k)$, we can see that

$$\upsilon_p = u_p \frac{\sigma_{pn}}{\sigma_{pk}} \quad \text{and} \quad \upsilon_n = \frac{\sigma_{nk}}{u_n}\,. \tag{B.4}$$

Similarly, we can match (5.3) with (5.14) to yield the expression for $\hat{u}_i$ and $\hat{u}_k$ using the same trick. The expression for $\tilde{\lambda}_n$ from $\tilde{\mathcal{E}}_k$ is substituted into $\tilde{\mathcal{E}}_i$ in (5.3), giving

$$\tilde{\mathcal{E}}_i^{\dot{\alpha}} = \sum_{p=k+1}^{n-1} \left(\frac{u_i u_p}{\sigma_{ip}} + \frac{u_i u_p \sigma_{kn}}{\sigma_{kp}\sigma_{in}} \right) \tilde{\kappa}_p^{\dot{\alpha}} - \tilde{\kappa}_i^{\dot{\alpha}} - \frac{\sigma_{kn} u_i}{\sigma_{in} u_k} \tilde{\kappa}_k \,. \tag{B.5}$$

Comparing the coefficient of $\tilde{\kappa}_k$ in (B.5) and (5.14), we obtain the relation

$$\frac{\sigma_{nk} u_i}{\sigma_{in} u_k} = \frac{\upsilon_i \upsilon_k}{\sigma_{ik}} \,. \tag{B.6}$$

Similarly, under the assumption that $\upsilon_i = \upsilon_i(u_i, \sigma_i, \sigma_n, \sigma_k)$ and $\upsilon_k = \upsilon_k(u_k, \sigma_n, \sigma_k)$, we can see that

$$\upsilon_i = u_i \frac{\sigma_{ik}}{\sigma_{in}} \quad \text{and} \quad \upsilon_k = \frac{\sigma_{nk}}{u_k} \,. \tag{B.7}$$

We can check that by using (B.4) and (B.7), one can transform (5.3) and (5.5) to (5.14) and (5.15). For example, one can transform (B.5) using (B.4) and (B.7) as the following:

$$\begin{aligned}
\tilde{\mathcal{E}}_i^{\dot{\alpha}} &= \sum_{p=k+1}^{n-1} \left(\frac{1}{\sigma_{ip}} + \frac{\sigma_{kn}}{\sigma_{kp}\sigma_{in}} \right) u_i u_p \tilde{\kappa}_p^{\dot{\alpha}} - \tilde{\kappa}_i^{\dot{\alpha}} - \frac{\sigma_{kn} u_i}{\sigma_{in} u_k} \tilde{\kappa}_k \\
&= \sum_{p=k+1}^{n-1} \left(\frac{\sigma_{np}\sigma_{ik}}{\sigma_{ip}\sigma_{kp}\sigma_{in}} \right) \left(\frac{\sigma_{in}}{\sigma_{ik}} \right) \left(\frac{\sigma_{pk}}{\sigma_{pn}} \right) \upsilon_i \upsilon_p \tilde{\kappa}_p^{\dot{\alpha}} - \tilde{\kappa}_i^{\dot{\alpha}} - \frac{\sigma_{kn}}{\sigma_{in}} \left(\frac{\sigma_{in}}{\sigma_{ik}} \right) \upsilon_i \frac{\upsilon_k}{\sigma_{nk}} \tilde{\kappa}_k \\
&= \sum_{p=k+1}^{n-1} \left(\frac{1}{\sigma_{ip}} \right) \upsilon_i \upsilon_p \tilde{\kappa}_p^{\dot{\alpha}} + \frac{\upsilon_i \upsilon_k}{\sigma_{ik}} \tilde{\kappa}_k^{\dot{\alpha}} - \tilde{\kappa}_i^{\dot{\alpha}} \,,
\end{aligned} \tag{B.8}$$

which is the constraint equation (5.14).

Coming back to out assumption before, we would not get the correct transformation if we had we had chosen the other choice for υs. This can be seen explicitly by having

$$\upsilon_p = u_p \frac{\sigma_{pn}\sigma_{nk}}{\sigma_{pk}} \qquad \text{and} \qquad \upsilon_n = \frac{1}{u_n} \,, \tag{B.9}$$

$$\upsilon_i = u_i \frac{\sigma_{ik}\sigma_{nk}}{\sigma_{in}} \qquad \text{and} \qquad \upsilon_k = \frac{1}{u_k} \,, \tag{B.10}$$

and substitute it into (B.5). This choice of variables would not give the correct expression for the transformed constraint equation in terms of υ:

$$\begin{aligned}
\tilde{\mathcal{E}}_i^{\dot{\alpha}} &= \sum_{p=k+1}^{n-1} \left(\frac{1}{\sigma_{ip}} + \frac{\sigma_{kn}}{\sigma_{kp}\sigma_{in}} \right) u_i u_p \tilde{\kappa}_p^{\dot{\alpha}} - \tilde{\kappa}_i^{\dot{\alpha}} - \frac{\sigma_{kn} u_i}{\sigma_{in} u_k} \tilde{\kappa}_k \\
&= \sum_{p=k+1}^{n-1} \left(\frac{\sigma_{np}\sigma_{ik}}{\sigma_{ip}\sigma_{kp}\sigma_{in}} \right) \left(\frac{\sigma_{in}}{\sigma_{ik}\sigma_{nk}} \right) \left(\frac{\sigma_{pk}}{\sigma_{pn}\sigma_{nk}} \right) \upsilon_i \upsilon_p \tilde{\kappa}_p^{\dot{\alpha}} - \tilde{\kappa}_i^{\dot{\alpha}} - \frac{\sigma_{kn}}{\sigma_{in}} \left(\frac{\sigma_{in}}{\sigma_{ik}\sigma_{nk}} \right) \upsilon_i \upsilon_k \tilde{\kappa}_k \\
&= \sum_{p=k+1}^{n-1} \left(\frac{1}{\sigma_{ip}\sigma_{nk}\sigma_{nk}} \right) \upsilon_i \upsilon_p \tilde{\kappa}_p^{\dot{\alpha}} - \tilde{\kappa}_i^{\dot{\alpha}} + \frac{\upsilon_i \upsilon_k}{\sigma_{ik}} \tilde{\kappa}_k \,,
\end{aligned}$$

which is not the transformed constraint equation as given by (5.14).

The reduced measure after the transformation is given by

$$\begin{aligned} d\mu'_{n,k} &= \frac{\prod_{i=1}^n d\sigma_i d\upsilon_i/\upsilon_i}{\text{vol}(SL(2,\mathbb{C})\times U(1))} \prod_{i=n}^{k-1} \bar{\delta}^2\left(\tilde{\kappa}_i - \upsilon_i \sum_{p=k}^{n-1} \frac{\upsilon_p \tilde{\kappa}_p}{\sigma_{ip}}\right) \prod_{p=k}^{n-1} \bar{\delta}^2\left(\kappa_p - \upsilon_p \sum_{i=n}^{k-1} \frac{\upsilon_i \kappa_i}{\sigma_{pi}}\right) \\ &= \frac{\prod_{i=1}^n d\sigma_i du_i/u_i}{\text{vol}(SL(2,\mathbb{C})\times U(1))} \prod_{i=1}^{k-1} \bar{\delta}^2\left(\tilde{\kappa}_i - u_i \sum_{p=k+1}^{n} \frac{u_p \tilde{\kappa}_p}{\sigma_{ip}}\right) \prod_{p=k+1}^{n-1} \bar{\delta}^2\left(\kappa_p - u_p \sum_{i=1}^{k} \frac{u_i \kappa_i}{\sigma_{pi}}\right) \\ &\quad \times \bar{\delta}^2\left(\tilde{\kappa}_n - \upsilon_n \sum_{p=k}^{n-1} \frac{\upsilon_p \tilde{\kappa}_p}{\sigma_{np}}\right) \bar{\delta}^2\left(\kappa_k - \upsilon_k \sum_{i=n}^{k-1} \frac{\upsilon_i \kappa_i}{\sigma_{ki}}\right) \end{aligned} \tag{B.11}$$

where we have used

$$\frac{d\upsilon_p}{\upsilon_p} = \frac{du_p}{u_p}, \qquad \frac{d\upsilon_i}{\upsilon_i} = \frac{du_i}{u_i}, \qquad \frac{d\upsilon_k}{\upsilon_k} = -\frac{du_k}{u_k}, \qquad \frac{d\upsilon_n}{\upsilon_n} = -\frac{du_n}{u_n}, \tag{B.12}$$

to show that $\prod_i d\upsilon_i/\upsilon_i = \prod_i du_i/u_i$. For the unswitched particle, the constraints remain the same under the inverse transformation as shown in (B.8). For particle k and n, the constraints can be transformed back as

$$\begin{aligned} \tilde{\kappa}_n - \upsilon_n \sum_{p=k}^{n-1} \frac{\upsilon_p \tilde{\kappa}_p}{\sigma_{np}} &= \tilde{\kappa}_n^{\dot{\alpha}} - \frac{\sigma_{nk}}{u_n} \sum_{p=k+1}^{n-1} \frac{u_p \tilde{\kappa}_p^{\dot{\alpha}}}{\sigma_{np}} \frac{\sigma_{pn}}{\sigma_{pk}} - \frac{\sigma_{nk}}{u_n} \frac{\tilde{\kappa}_k^{\dot{\alpha}}}{\sigma_{nk}} \frac{\sigma_{nk}}{u_k} \\ &= u_n u_k \frac{\tilde{\kappa}_n^{\dot{\alpha}}}{\sigma_{nk}} + \sum_{p=k+1}^{n-1} \frac{u_p u_k \tilde{\kappa}_p^{\dot{\alpha}}}{\sigma_{pk}} - \tilde{\kappa}_k^{\dot{\alpha}} = u_k \sum_{p=k+1}^{n} \frac{u_p \tilde{\kappa}_p^{\dot{\alpha}}}{\sigma_{kp}} - \tilde{\kappa}_k^{\dot{\alpha}}, \end{aligned} \tag{B.13}$$

and

$$\kappa_k - \upsilon_k \sum_{i=n}^{k-1} \frac{\upsilon_i \kappa_i^{\alpha}}{\sigma_{ki}} = \kappa_k^{\alpha} - \frac{\sigma_{nk}}{u_k} \sum_{i=1}^{k-1} \frac{\kappa_i^{\alpha}}{\sigma_{ki}} u_i \frac{\sigma_{ik}}{\sigma_{in}} - \frac{\sigma_{nk}}{u_k} \frac{\sigma_{nk}}{u_n} \frac{\kappa_n^{\alpha}}{\sigma_{kn}} = u_n \sum_{i=1}^{k} \frac{u_i \kappa_i^{\alpha}}{\sigma_{ni}} - \kappa_n^{\alpha}. \tag{B.14}$$

Thus, the reduced measure under the transformation is equal to the original reduced measure up to a Jacobian factor:

$$d\mu'_{n,k} = d\mu_{n,k} \left(\frac{\sigma_{nk}}{u_n u_k}\right)^4, \tag{B.15}$$

which would cancel out with the transformation factors of the Grassmannian delta functions in $\mathcal{N} = 4$ SYM.

Therefore, the reduced measure is invariant of the choice of helicity for each particle and depends only on the number of particles in each sectors. □

Appendix C

Supersymmetric Amplitudes

RSVW formula was originally derived from a maximally supersymmetric string theory with twistor space as its target space, called the twistor string theory.[1] Even though twistor string and the derivation of RSVW formula is beyond the scope of this project, we will discuss the supersymmetric amplitudes in this appendix for completeness.

It is also crucial to point out the this project's main focus is restricted only to the bosonic sector i.e. the amplitude with gluons as external particles. The reason that we can focus only on the bosonic sector is that at the tree level, there are no fermionic contribution because the fermion vertex requires the fermion to come in and out. This means that an amplitude with bosons as the external particles will contain fermions only at the loop level. [2] Nevertheless, it is useful to remark on what superamplitudes are, to gain more insights on MHV and NMHV and to prepare for the RSVW formula that will be presented in chapter 4.

The $\mathcal{N} = 4$ super Yang-Mills theory massless supermultiplet consists of 16 states: a gluon of positive helicity g^+, 4 gluinos λ^A with helicity $1/2$, 6 scalars S^{AB}, 4 gluinos λ^{ABC} with helicity $-1/2$, and a gluon of negative helicity g^-.[3] Solving the field equations and quantizing the field, each field can be represented by the creation-annihilation operators. Since we are looking at the scattering with all particles outgoing, these states can be represent by its annihilation operators. Here, they are denoted a for the positive helicity gluon, a^A for the positive helicity gluinos, a^{AB} for the scalars, a^{ABC} for the negative helicity gluinos, and a^{1234} for the negative helicity gluon, with the indices $A, B = 1, .., 4$. These indices serve purpose as the label of the $SU(4)$ symmetry that rotates the supersymmetry generators Q^A and the conjugate $\tilde{Q}_A \equiv Q^\dagger_A$. The action of the supercharges on the

[1] In fact, twistor string theory is not well-defined for theories with less than maximal supersymmetry [5].

[2] Same argument holds with the CHY formula: one can restrict to the bosonic sector after deriving the formula the RNS ambitwistor string theory, that is not well-defined for less than maximal supersymmetry.

[3] The gluinos are not to be confused with the spinor λ_α with the greek alphabet index, though, the gluinos will only appear in this section.

annihilation operators is

$$\begin{aligned}
[\tilde{Q}_{A\alpha}, a(\kappa)] &= 0 & [Q^A_{\dot{\alpha}}, a(\kappa)] &= \tilde{\kappa}_{\dot{\alpha}} a^A(\kappa) \\
[\tilde{Q}_{A\alpha}, a^B(\kappa)] &= \kappa_\alpha \delta^B_A a(\kappa) & [Q^A_{\dot{\alpha}}, a^B(\kappa)] &= \tilde{\kappa}_{\dot{\alpha}} a^{AB}(\kappa) \\
[\tilde{Q}_{A\alpha}, a^{BC}(\kappa)] &= \kappa_\alpha 2! \delta^{[B}_A a^{c]}(\kappa) & [Q^A_{\dot{\alpha}}, a^{BC}(\kappa)] &= \tilde{\kappa}_{\dot{\alpha}} a^{ABC}(\kappa) \\
[\tilde{Q}_{A\alpha}, a^{BCD}(\kappa)] &= \kappa_\alpha 3! \delta^{[B}_A a^{CD]}(\kappa) & [Q^A_{\dot{\alpha}}, a^{BCD}(\kappa)] &= \tilde{\kappa}_{\dot{\alpha}} a^{ABCD}(\kappa) \\
[\tilde{Q}_{A\alpha}, a^{BCDE}(\kappa)] &= \kappa_\alpha 4! \delta^{[B}_A a^{CDE]}(\kappa) & [Q^A_{\dot{\alpha}}, a^{1234}(\kappa)] &= 0
\end{aligned} \tag{C.1}$$

where the κ and $\tilde{\kappa}$ in the operators denotes the spinors of momentum of the outgoing state, and the bracket is an anticommutator if both the arguments are fermionic and otherwise a commutator.

We can now look at some consequences of SUSY on studying the amplitude. An n point amplitude can be written using the annihilation operators for each particle, $\mathcal{O}_i(p_i)$, with i being the particle label, acting to the left on the out-vaccuum:

$$\langle 0| \mathcal{O}_1(p_1)...\mathcal{O}_n(p_n) |0\rangle \ . \tag{C.2}$$

Supposing the vaccuum is supersymmetric, the vaccuum must then be annihilated by the SUSY generators: $Q^A|0\rangle = 0 = \tilde{Q}_A|0\rangle$. Generators annihilating the vacuum means that

$$0 = \langle 0| Q^A \mathcal{O}_1(p_1)...\mathcal{O}_n(p_n) |0\rangle = \langle 0| \mathcal{O}_1(p_1)...\mathcal{O}_n(p_n) Q^A |0\rangle = \langle 0| \left[Q^A, \mathcal{O}_1(p_1)...\mathcal{O}_n(p_n)\right] |0\rangle \ , \tag{C.3}$$

and the same holds for $\tilde{Q}_A$. These relations are called supersymmetric Ward identities, and these allow us to write a linear relation among the scattering amplitudes with the external states related by supersymmetry:

$$\begin{aligned}
0 &= \langle 0| \left[Q^A, \mathcal{O}_1(p_1)...\mathcal{O}_n(p_n)\right] |0\rangle \\
&= \sum_{i=1}^{n} (-1)^{\sum_{i<j} |\mathcal{O}_j|} \langle 0| \mathcal{O}_1(p_1)... \left[Q^A, \mathcal{O}_i(p_i)...\mathcal{O}_n(p_n)\right] |0\rangle \ .
\end{aligned} \tag{C.4}$$

The supersymmetric Ward identities can be used to show that the all plus/minus amplitude vanishes not only at tree level but all loop level. Consider

$$0 = \langle 0| \left[\tilde{Q}_A, a^B_1(p_1) a_2(p_2)...a_n(p_n)\right] |0\rangle \ , \tag{C.5}$$

using the commutation relations (C.1), one finds that

$$0 = p_{\dot{\alpha}} \delta^B_A \langle 0| a_1(p_1) a_2(p_2)...a_n(p_n) |0\rangle = p_{\dot{\alpha}} \delta^B_A A_n[+++...+] \, , \tag{C.6}$$

the amplitude with all-plus external gluons vanishes. We can use the same method to show that the one minus/plus amplitude also vanishes. This holds true at all orders in perturbation theory since it relies on the fact that the vacuum is annihilated by the supersymmetric generators.

Extending to the on-shell superspace, we can collect the states in the supermultiplet into a superfield by introducing four Grassmannian variables η labeled by $SU(4)$ index:

$$\Omega = g^+ + \eta_A \lambda^A - \frac{1}{2!} \eta_A \eta_B S^{AB} - \frac{1}{3!} \eta_A \eta_B \eta_C \lambda^{ABC} + \eta_1 \eta_2 \eta_3 \eta_4 g^- \, , \tag{C.7}$$

where the signs are chosen such that the Grassmannian differential operators will select its associate states from the superfield. The Grassmannian differential operators for the i^{th} particle are

$$1,\ \frac{\partial}{\partial\eta_{iA}},\ \frac{\partial}{\partial\eta_{iA}}\frac{\partial}{\partial\eta_{iB}},\ \frac{\partial}{\partial\eta_{iA}}\frac{\partial}{\partial\eta_{iB}}\frac{\partial}{\partial\eta_{iC}},\ \text{and}\ \frac{\partial}{\partial\eta_{i1}}\frac{\partial}{\partial\eta_{i2}}\frac{\partial}{\partial\eta_{i3}}\frac{\partial}{\partial\eta_{i4}}\,, \tag{C.8}$$

associating with g^+, λ^A, S^{AB}, λ^{ABC}, and g^-, respectively. This allow us to define the supercharges as

$$q^{A\dot{\alpha}} := \tilde{\kappa}^{\dot{\alpha}}\frac{\partial}{\partial\eta_A} \quad \text{and} \quad q_A^{\dagger\alpha} := \kappa^\alpha\eta_A\,, \tag{C.9}$$

where the spinors κ_α and $\tilde{\kappa}_{\dot{\alpha}}$ are the spinors associated with the momentum of the particle. The supercharges satisfies $\{q^{A\dot{\alpha}}, q_B^{\dagger\alpha}\} = \delta_B^A\kappa^\alpha\tilde{\kappa}^{\dot{\alpha}}$.
With the superfield, we can write a superamplitude $\mathcal{A}_n[\Omega_1...\Omega_n]$, which depends on the on-shell momentum p_i and a set of Grassmann variable η_{iA} for each particle $i = 1, ..., n$. Using the differential operators (or equivalently in the form of Grassmann integrals), we can extract the states from the superfields and hence extract the amplitudes from the superamplitudes. The supersymmetric Ward identity for on-shell superspace is now

$$\mathcal{Q}^A\mathcal{A}_n = 0 \quad \text{and} \quad \tilde{\mathcal{Q}}_A\mathcal{A}_n = 0\,, \tag{C.10}$$

where

$$\mathcal{Q}^{A\dot{\alpha}} := \sum_{i=1}^{n} q_i^{A\dot{\alpha}} \quad \text{and} \quad \tilde{\mathcal{Q}}_A^{\alpha} := \sum_{i=1}^{n} q_{iA}^{\dagger\alpha}\,. \tag{C.11}$$

Using the fact that $\mathcal{Q}^A$ and $\tilde{\mathcal{Q}}_A$ annihilates the superamplitude, we know that the total momentum $\sum_{i=1}^n p_i \sim \{\mathcal{Q}^A, \tilde{\mathcal{Q}}_A\}$ annihilates the amplitude as well. This is actually the momentum conservation in an operational form. Using the Grassmannian delta function

$$\delta^8(\tilde{\mathcal{Q}}) = \frac{1}{2^4}\prod_{A=1}^{4}\tilde{\mathcal{Q}}_{A\alpha}\tilde{\mathcal{Q}}_A^{\alpha} = \frac{1}{2^4}\prod_{A=1}^{4}\sum_{i,j=1}^{n}\langle ij\rangle\,\eta_{iA}\eta_{jA}\,. \tag{C.12}$$

the superamplitude can be written as

$$\mathcal{A}_n = \delta^8(\tilde{\mathcal{Q}})P_n\,, \tag{C.13}$$

where P_n is some polynomial of Grassmann variables of degree $4k$, where k is the number of negative helicity gluons [28]. The Grassmannian delta function guarantees the annihilation of the superamplitude by the supercharges: obviously $\tilde{\mathcal{Q}_A}\delta^8(\tilde{\mathcal{Q}_A}) = 0$ and $\mathcal{Q}^A\delta^8(\tilde{\mathcal{Q}_A}) = 0$ by momentum conservation. The tree-level MHV superamplitude of $\mathcal{N} = 4$ SYM can be written as

$$\mathcal{A}_n^{\text{MHV}}[123...n] = \frac{\delta^8(\tilde{\mathcal{Q}})}{\langle 12\rangle\langle 23\rangle ... \langle n1\rangle}\,, \tag{C.14}$$

in which the Parke-Taylor tree-amplitude can be obtained by applying the four derivatives withe respect to η_{iA} and η_{jA} for the negative helicity particle i and j, then the delta function would return the factor $\langle ij\rangle^4$.

Appendix D

Spinors in Six Dimensions

The off-shell formalism in [14] was done first in six dimensions, so we will mention the formulation of spinors in six dimensions. We follow the construction of six dimensions spinors in [29]. In six dimensions, the Lorentz group is $SO(6) \simeq SU(4)$, the spinors are then complex four component objects transforming under fundamental representation of $SU(4)$. We expect the 6d momentum to be written as some antisymmetric product of two spinors, since the antisymmetric representation of $SU(4)$ is six dimensional. The gamma matrices of this representation are given by γ^{μ}_{AB} and $\tilde{\gamma}^{\mu,AB}$, with $A, B = 1, ..., 4$, where the two gamma matrices are the fundamental and anti-fundamental representation of $SU(4)$. But unlike the usual $SU(2)$ where we can use the invariant tensor $\epsilon_{\alpha\beta}$ to raise and lower an index, the fundamental and anti-fundamental cannot be raised or lowered like that. The only invariant tensor of $SU(4)$ is ϵ_{ABCD}, so it will raise amd lower both indices:

$$\epsilon_{ABCD}\gamma^{CD} = \gamma_{AB} \quad \text{and} \quad \epsilon^{ABCD}\gamma_{CD} = \gamma^{AB} \,. \tag{D.1}$$

The little group in 6d is the $SO(4) \simeq SU(2) \times SU(2)$, and they are labeled by $a = 1, 2$ and $\dot{a} = 1, 2$, respectively.

The 6d massless momentum k_μ in the spinor notation looks like

$$k_{AB} := k_\mu \gamma^{\mu}_{AB} \quad \text{and} \quad \tilde{k}^{AB} := k_\mu \tilde{\gamma}^{\mu,AB} \,. \tag{D.2}$$

Since the index A and B is antisymmetric, we can decompose these into an antisymmetric product of spinors with a little group index,

$$k^{AB} = \varkappa_1^A \varkappa_2^B - \varkappa_2^A \varkappa_1^B = \varkappa_a^A \varkappa_b^B \epsilon^{ab} := (\!(\varkappa^A \varkappa^B)\!) \tag{D.3}$$

$$\tilde{k}_{AB} = \tilde{\varkappa}_{1A}\tilde{\varkappa}_{2B} - \tilde{\varkappa}_{2A}\tilde{\varkappa}_{1B} = \tilde{\varkappa}_{\dot{a}A}\tilde{\varkappa}_{\dot{b}B}\epsilon^{\dot{a}\dot{b}} := (\!(\tilde{\varkappa}_A \tilde{\varkappa}_B)\!) \,, \tag{D.4}$$

for some 6d spinors $\varkappa_a^A$ and $\tilde{\varkappa}_{aA}$. The double round brackets will denote the contraction of the 6d little group index.

Now, we need to embed what we had in 4d in to this 6d notation. The 4d momentum is embedded in a 6d spinor as

$$\varkappa_a^A = \begin{pmatrix} \kappa_1^{\alpha} & \kappa_2^{\alpha} \\ \tilde{\kappa}_1^{\dot{\alpha}} & \tilde{\kappa}_2^{\dot{\alpha}} \end{pmatrix} \,, \tag{D.5}$$

where the κ^α and $\tilde{\kappa}^{\dot{\alpha}}$ are the usual 4d spinors. One can see that via this construction, the massless 4d momentum can be written as

$$\varkappa_a^A = \begin{pmatrix} 0 & \kappa^\alpha \\ \tilde{\kappa}^{\dot{\alpha}} & 0 \end{pmatrix} , \tag{D.6}$$

such that $k^{AB} = \varkappa_a^A \varkappa_b^B \epsilon^{ab}$ reduces to $\kappa^\alpha \tilde{\kappa}^{\dot{\alpha}} = k^{\alpha\dot{\alpha}}$. The 4d off-shell momentum l, on the other hand, is embedded in 6d as

$$l^{\alpha\dot{\alpha}} = \kappa_1^\alpha \tilde{\kappa}_2^{\dot{\alpha}} - \kappa_2^\alpha \tilde{\kappa}_1^{\dot{\alpha}} . \tag{D.7}$$

For further convenience, we want to stick with the 4d notation, we split the 6d spinor into two parts:

$$\kappa_\alpha^a = \left(\kappa_\alpha^1, \kappa_\alpha^2\right) \quad \text{and} \quad \tilde{\kappa}_{\dot{\alpha}}^a = \left(\tilde{\kappa}_\alpha^1, \kappa_\alpha^2\right) , \tag{D.8}$$

so that the massive momentum l with lowered indices can be expressed as

$$l_{\alpha\dot{\alpha}} = \kappa_\alpha^a \tilde{\kappa}_{\dot{\alpha}}^b \epsilon_{ab} , \tag{D.9}$$

with

$$\kappa_\alpha^a \kappa_\beta^b \epsilon_{ab} = M \epsilon_{\alpha\beta} \quad \text{and} \quad \tilde{\kappa}_{\dot{\alpha}}^a \tilde{\kappa}_{\dot{\beta}}^b \epsilon_{ab} = \tilde{M} \epsilon_{\dot{\alpha}\dot{\beta}} , \tag{D.10}$$

such that $l^2 = M\tilde{M}$.

In [14], the loop amplitude in 4d can be obtained by the 6d amplitude with the loop momenta l flowing through the two nodal points l_+ and l_- on the forward limit (like the back-to-back limit in chapter 6). The forward limit requires the 6d spinors of the off-shell loop momenta to be

$$\varkappa_{+,a}^A = \begin{pmatrix} \kappa_1^\alpha & \kappa_2^\alpha \\ \tilde{\kappa}_1^{\dot{\alpha}} & \tilde{\kappa}_2^{\dot{\alpha}} \end{pmatrix} \quad \text{and} \quad \varkappa_{-,a}^A = \begin{pmatrix} \kappa_1^\alpha & -\kappa_2^\alpha \\ \tilde{\kappa}_1^{\dot{\alpha}} & -\tilde{\kappa}_2^{\dot{\alpha}} \end{pmatrix} , \tag{D.11}$$

such that $((\varkappa_+^A \varkappa_+^B)) = -((\varkappa_-^A \varkappa_-^B))$. Splitting the 6d spinors into two parts, we obtain

$$\kappa_{+,\alpha}^a = \left(\kappa_{+,\alpha}^1, \kappa_{+,\alpha}^2\right) \qquad \tilde{\kappa}_{+,\dot{\alpha}}^a = \left(\tilde{\kappa}_{+,\dot{\alpha}}^1, \tilde{\kappa}_{+,\dot{\alpha}}^2\right) , \tag{D.12}$$

$$\kappa_{-,\alpha}^a = \left(\kappa_{+,\alpha}^1, -\kappa_{+,\alpha}^2\right) \qquad \tilde{\kappa}_{-,\dot{\alpha}}^a = \left(\tilde{\kappa}_{+,\dot{\alpha}}^1, -\tilde{\kappa}_{+,\dot{\alpha}}^2\right) . \tag{D.13}$$

Bibliography

[1] M. D. Schwartz, *Quantum field theory and the standard model*. Cambridge Univ. Press, Cambridge, 2014, 1107034736.

[2] H. Elvang and Y.-t. Huang, *Scattering Amplitudes in Gauge Theory and Gravity*. Cambridge University Press, 4, 2015.

[3] S. J. Parke and T. R. Taylor, *An Amplitude for n Gluon Scattering*, *Phys. Rev. Lett.* **56** (1986) 2459.

[4] L. J. Dixon, *A brief introduction to modern amplitude methods*, in *Theoretical Advanced Study Institute in Elementary Particle Physics: Particle Physics: The Higgs Boson and Beyond*, 10, 2013, 1310.5353, DOI.

[5] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, *Commun. Math. Phys.* **252** (2004) 189–258, [hep-th/0312171].

[6] R. Roiban, M. Spradlin and A. Volovich, *A Googly amplitude from the B model in twistor space*, *JHEP* **04** (2004) 012, [hep-th/0402016].

[7] M. Atiyah, M. Dunajski and L. Mason, *Twistor theory at fifty: from contour integrals to twistor strings*, *Proc. Roy. Soc. Lond. A* **473** (2017) 20170530, [1704.07464].

[8] F. Cachazo and Y. Geyer, *A 'Twistor String' Inspired Formula For Tree-Level Scattering Amplitudes in N=8 SUGRA*, 1206.6511.

[9] L. Dolan and P. Goddard, *Tree and Loop Amplitudes in Open Twistor String Theory*, *JHEP* **06** (2007) 005, [hep-th/0703054].

[10] F. Cachazo, S. He and E. Y. Yuan, *Scattering of Massless Particles in Arbitrary Dimensions*, *Phys. Rev. Lett.* **113** (2014) 171601, [1307.2199].

[11] Y. Geyer, L. Mason, R. Monteiro and P. Tourkine, *Loop Integrands for Scattering Amplitudes from the Riemann Sphere*, *Phys. Rev. Lett.* **115** (2015) 121603, [1507.00321].

[12] S. He and E. Y. Yuan, *One-loop Scattering Equations and Amplitudes from Forward Limit*, *Phys. Rev. D* **92** (2015) 105004, [1508.06027].

[13] F. Cachazo, S. He and E. Y. Yuan, *One-Loop Corrections from Higher Dimensional Tree Amplitudes*, *JHEP* **08** (2016) 008, [1512.05001].

[14] C. Wen and S.-Q. Zhang, *D3-Brane Loop Amplitudes from M5-Brane Tree Amplitudes*, *JHEP* **07** (2020) 098, [2004.02735].

[15] M. Srednicki, *Quantum Field Theory*. Cambridge Univ. Press, Cambridge, 2007.

[16] T. Adamo, *Lectures on twistor theory*, 2018.

[17] G. A. Jones and D. Singerman, *Complex Functions: An Algebraic and Geometric Viewpoint*. Cambridge University Press, 1987, 10.1017/CBO9781139171915.

[18] N. Berkovits, *An Alternative string theory in twistor space for N=4 superYang-Mills*, *Phys. Rev. Lett.* **93** (2004) 011601, [hep-th/0402045].

[19] F. Cachazo, L. Mason and D. Skinner, *Gravity in Twistor Space and its Grassmannian Formulation*, *SIGMA* **10** (2014) 051, [1207.4712].

[20] Y. Geyer, *Ambitwistor Strings: Worldsheet Approaches to perturbative Quantum Field Theories*, Ph.D. thesis, Oxford U., Inst. Math., 2016. 1610.04525.

[21] F. Cachazo, S. Mizera and G. Zhang, *Scattering Equations: Real Solutions and Particles on a Line*, *JHEP* **03** (2017) 151, [1609.00008].

[22] T. K. Petersen, *Eulerian numbers*. Birkhäuser, 2015.

[23] K. A. Roehrig, *Chiral splitting and $\mathcal{N}=4$ Einstein-Yang-Mills tree amplitudes in 4d*, *JHEP* **08** (2017) 033, [1705.09315].

[24] Y. Zhang, *CHY formulae in 4d*, *JHEP* **07** (2017) 069, [1610.05205].

[25] G. Albonico, Y. Geyer and L. Mason, *Recursion and worldsheet formulae for 6d superamplitudes*, *JHEP* **08** (2020) 066, [2001.05928].

[26] Y. Geyer and L. Mason, *Polarized Scattering Equations for 6D Superamplitudes*, *Phys. Rev. Lett.* **122** (2019) 101601, [1812.05548].

[27] J. Hunter, *Asymptotic analysis and singular perturbation theory*, .

[28] H. Elvang, D. Z. Freedman and M. Kiermaier, *SUSY Ward identities, Superamplitudes, and Counterterms*, *J. Phys. A* **44** (2011) 454009, [1012.3401].

[29] C. Cheung and D. O'Connell, *Amplitudes and Spinor-Helicity in Six Dimensions*, *JHEP* **07** (2009) 075, [0902.0981].