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Reciprocity Principle and Extended Special Relativity

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Reciprocity Principle
and
Extended Special Relativity

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Abstract

Extended Lorentz transformations (ELT) based on spatial anisotropy and time isotropy of a particular inertial frame together with the reciprocity principle are derived. The isolated pair of two inertial frames is proposed to satisfy the condition of frame-independent relative velocities mapping. The relations of sub-superluminal spaces are discussed via continuous bijective mappings and the Einstein velocities addition formula. By using a relative velocity of two inertial frames as a parameter, which is similar to Hill-Cox [2] derivations, the consistent form of ELT is obtained. We then propose the transformation of the inertial frames' configuration by utilizing the three velocity mappings. By focusing on the issue of inconsistency with special relativity of the derived ELT, we further review the six-dimensional structure of spacetime which is suggested previously by an author [1]. In addition, we suggest a particular mathematical structure of spacetime, i.e., an ultra-hyperbolic (3, *n*)- structure for any finite $n \in \mathbb{Z}^+, n \geq 2$, under the assumption that only when an observation of light is made in either frame (of reference), an observer in that frame can access all *n*-degrees of freedom of time coordinates to maintain the spherical propagation of light in spatial dimensions (3D).

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List of Abbreviations

SR: Special Relativity

ESR: Extended Special Relativity

LT: Lorentz Transformations

ELT: Extended Lorentz Transformations

FTL: Faster Than Light

Chapter 1
Introduction

1.1 Background

Special relativity was historically formulated by Albert Einstein in 1905. This theory is based on two postulates, (1.) The principle of relativity and (2.) The constancy of the speed of light. The first postulate is a well-known principle about the equivalence of inertial frames of reference and invariance of the physical laws. The second postulate leads to many different perspectives compared to classical physics. However, this postulate was proposed to make Maxwell's equations invariant under coordinate transformations of any two inertial frames, which finally leads to the relative speed limit of any two inertial frames less than the speed of light (the possibility of the existence of superluminal particles is not restricted by the theory directly). There're many suggested formulations to extend special relativity to the superluminal case. In Vieira's paper [1], both algebraic and geometrical deductions of the extended Lorentz transformation (ELT) were introduced. The results of superluminal transformations in two dimensions (one spatial coordinate and one time coordinate) are similar to the Hill-Cox's transformations [2]. However, these particular transformations in the usual four-dimensional spacetime cause some problems, for instance, inconsistency with the principle of relativity as stated in [3] and invalidity of the principle of the speed of light invariance as claimed by [1]. Some solutions to the so-called problems are suggested, namely, by introducing imaginary coordinates to the inertial frame's transformations [1] or by considering the six-real dimensional spacetime (three spatial coordinates and three-time coordinates) with some suitable conditions [1, 4]. This present project provides the derivations of the ELT for the superluminal motion based on the reciprocity principle [7], and by using a relative velocity of two inertial frames as a parameter, which is similar to Hill-Cox [2] derivations, the consistent form of ELT is obtained, which is compatible with the result shown in [5]. The main derivations follow the steps in [7] by using the consequence of the homogeneity of space-time that implies the inertial frames linear transformation presented in [7] together with the conditions that space is not isotropic and time is isotropic. A further crucial step is to interpret the obtained results which some authors are concerned about their validity. By reviewing the provided imaginary coordinates and six-dimensional spacetime scenarios, some geometrical interpretations are suggested. Two common consequences of the previously suggested ELT are anisotropy of space and the universality of velocities addition law. According to the second consequence, the relations of subluminal and superluminal frames are discussed [1, 2, 8] via continuous bijective mappings. The ultra-hyperbolic (3, 3)-structure of spacetime suggested previously by an author [1] is discussed in detail. By using the (interchangeably) reciprocal relations of velocities as presented in [8] and the obtained ELT, we then propose the transformation of the inertial frames' configuration by utilizing the three velocity mappings which are continuous on their domains, and they are also bijective. Besides, we suggest a particular mathematical structure of spacetime, i.e., an ultra-hyperbolic (3, *n*)- structure for any finite $n \in \mathbb{Z}^+, n \geq 2$, under the assumption that only when an observation of light is made in either frame (of reference), an observer in that frame can access all *n*-degrees of freedom of time coordinates to maintain the spherical propagation of light in spatial dimensions (3D).

1.2 Objectives

1. To derive the Extended Lorentz Transformations (ELT) based on the symmetries of spacetime together with the reciprocity principle and modify the Hill-Cox transformations.
2. To study the geometry of spacetime associated with the derived ELT and review the previously proposed 6-dimensional spacetime.
3. To propose the transformation of the inertial frames' configuration which corresponds to properties of velocities mappings.
4. To suggest an ultra-hyperbolic (3, *n*)- structure of spacetime to maintain the principle of invariance of the speed of light associated with the ELT.

1.3 Tools

1. Reference papers
2. A personal computer

Chapter 2

Literature Review and Theoretical Framework

2.1 Review of Special Relativity

In 1905, Albert Einstein published many great legendary papers, one of them is about the theory that revolutionized the fundamental aspects of nature, i.e., space and time. Later time, the particular theory is called the "special theory of relativity" or "SR" for short. The original paper [6], "on the electrodynamics of moving bodies", was mainly about how observers being in different inertial frames measure the same events would agree with the same physical law-electromagnetism. Einstein pointed out the effect of simultaneity of events and the way to synchronize any two clocks of two inertial observers. Special relativity is based on two postulates, first is the principle of relativity which can be classified into two parts, i.e., the weaker version and the stronger version. The weaker version is due to the Newtonian mechanics, namely, the first Newton's law of motion, by applying the system paradigm, force is defined as the influence that causes an object's motion and object contains a physical quantity called mass which resists the change of its temporary state, velocity, by an external force. Newton regards that the universe has an absolute time, which roles as a dynamical parameter and runs with the same rate for any observer, and it's also independent of space that is considered as the background for events to take place. And both space and time can be coordinatized by each observer. The bodies (point particles) exerted by zero net force must travel in the uniform linear motion with constant velocities. Thus, there exists, in principle, a privileged set of those bodies. The frame which moves along one of those bodies (co-moving observer) is called an inertial frame (of reference). Since the existence of an inertial frame is a consequence of Newton's laws of motion, it is equivalent to say that Newton's laws of motion are valid in all inertial frames. The restricted principle of special relativity extends the concept of a weaker version with an assumption that physical laws (not only Newton's laws of motion) should be universally invariant for all observers attached to inertial frames. This stronger version includes the law of electromagnetism, this is the starting point of the second postulate of special relativity which states that the speed of light is invariant for all inertial observers. According to these two postulates, there are several peculiar consequences from the theory such as time dilation, length contraction and equivalence of mass and energy which is denoted in the famous formula, $E = mc^2$. The remarkable mathematical structure of special relativity is the Lorentz Transformations, which was introduced by Hendrik Lorentz to explain the results of the Michelson-Morley experiment before the special relativity was completely formulated. Moreover, the geometry of spacetime with Lorentz signature was proposed by a mathematician, Hermann Minkowski, this structure of spacetime is called Minkowski space. Minkowski space is a Pseudo-Euclidean space together with the quadratic form classified as the hyperbolic equation.

One crucial aspect of this particular hyperbolic structure of spacetime is that a metric (tensor) is the same regardless of any inertial frames, which is different from the more general theory, i.e., general relativity. Notice that the word "spacetime" is used instead of "space-time", in special relativity, space and time are combined to be one mathematical structure, which is different from "Newtonian space-time", except when one refers to space-time transformations to focus on a particular coordinate.

For simplicity, we will consider two-dimensional space-time transformations which conserve the space-time origin. This special case can always be possibly assumed, since any transformation can always be reduced to a homogeneous velocity transformation along a particular axis by choosing a suitable space-time translation, together with suitable rotations of the spatial axes of the two observers, as stated in [7] though out this report.

2.1.1 Review of Lorentz Transformations

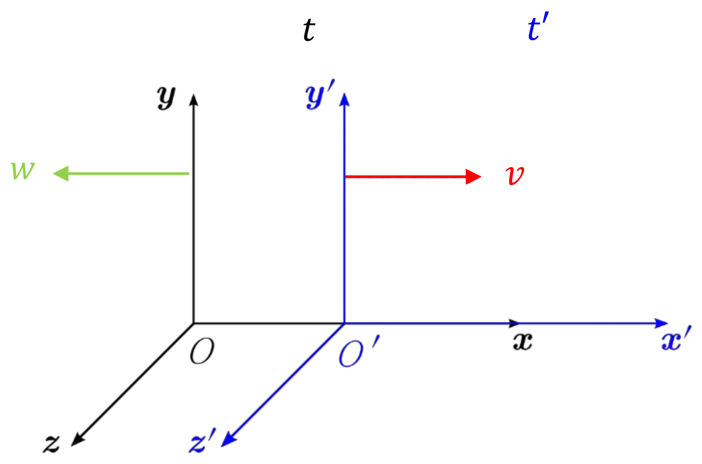

Figure 2.1: Two inertial frames moving along the x-direction with the relative velocity v.

Suppose we consider a relative motion of a pair of inertial frames such that Cartesian Coordinates are applied to both frames. Let the first frame has its coordinates as (x, y, z) and the second frame has (x', y', z') together with their times coordinates t and t' respectively, see **Figure 2.1**. Let call these two frames as S and S', and their relative motion is along the x and x'axes with the relative velocity of S' with respect to S as v.

Let call the relative velocity of S' with respect to S as "direct velocity" and the relative velocity of S with respect to S' as "reciprocal velocity". We further assume the reciprocity principle in this case, i.e., the reciprocal velocity is $-v$. Although this is a physically reasonable assumption, we will further discuss this principle in detail and one will see that this principle is not a trivial condition to assume.

The following brief discussions of the Lorentz transformations and the hyperbolicity of the Minkowski spacetime will be similar to the steps presented in [2].

For $0 \le |v| < c$, the usual Lorentz transformations are

$$x = \frac{x'+vt'}{\sqrt{1-(v/c)^2}}, \qquad t = \frac{t'+vx'/c^2}{\sqrt{1-(v/c)^2}}.$$

With the inverse transformations characterized by the reciprocal velocity $-v$, that is

$$x' = \frac{x-vt}{\sqrt{1-(v/c)^2}}, \qquad t = \frac{t'+vx'/c^2}{\sqrt{1-(v/c)^2}}. \tag{2.1.1}$$

The transformations above show that the two origin of two frames coincide when the relative velocity is zero, namely

$$x' = x,\ t' = t,\ v = 0. \tag{2.1.2}$$

Obviously see that

$$x + ct = x' = ct',\ x - ct = x' - ct',\ v = 0. \tag{2.1.3}$$

We can view equation (2.1.2) as a prescribed constraint when the parameter v is zero. Equations (2.1.2) and (2.1.3) explain the same physical event as viewed from two distinct moving inertial frames with $0 \le |v| < c$.

From equation (2.1.1),

$$x'+ct' = \left(\frac{1-v/c}{1+v/c}\right)^{1/2}(x+ct), \qquad x'-ct' = \left(\frac{1+v/c}{1-v/c}\right)^{1/2}(x-ct). \tag{2.1.4}$$

For $v = 0$, they reduce to (2.1.3). One can further deduce the Lorentz invariance equation,

$$x^2 - (ct)^2 = x'^2 - (ct')^2. \tag{2.1.5}$$

Recall the hyperbolic identity $\cosh^2 x - \sinh^2 x = 1$, from (2.1.1) we get

$$\cosh\phi = \frac{1}{\sqrt{1-(v/c)^2}}, \text{ and } \sinh\varphi = \frac{v/c}{\sqrt{1-(v/c)^2}}. \tag{2.1.6}$$

(2.1.5) reflects that spacetime together with Lorentz transformations is hyperbolic, called Minkowski space.

By writing $u = \frac{dx}{dt}$ and $u' = \frac{dx'}{dt'}$, then substitute in (2.1.4), one obtains

$$\frac{u'+c}{u'-c} = \left(\frac{1-v/c}{1+v/c}\right)\left(\frac{u+c}{u-c}\right). \qquad (2.1.7)$$

As an equivalent equation, then velocities addition formula can be deduced.

$$u' = \frac{u-v}{1-uv/c^2}. \qquad (2.1.8)$$

2.1.2 Symmetries of Space and Time, Reciprocity Principle and the Lorentz Transformations

We start from the following assumptions[1]:
1.) The principle of relativity, which implies the equivalence of all inertial frames to formulate the same physical laws of nature.
2.) The homogeneity of space and time
3.) The isotropy of space

We denote the position at which an event takes place with x and by t the time at which it occurs as observed by an inertial frame S, and the same event is described by x' and t' as viewed by S'.

Let $\xi = (x,t)$ and $\xi' = (x',t')$. Define a transformation $f : \xi \rightarrow \xi'$ as

$$\xi' = f(\xi). \qquad (2.1.9)$$

Suppose that this transformation conserves the space-time origin, it requires that $f(0) = 0$. For the general case, this condition is not necessary.

The homogeneity of space-time requires that a space-time translation T does not affect the relations between the two observers and leaves (2.1.9) invariant. Denote T_α and $T_{\alpha'}$ as the space-time translation relative to S and S' respectively. This property can be expressed by the relation

$$f(T_\alpha \xi) = T_{\alpha'} f(\xi). \qquad (2.1.10)$$

One may write the equivalent relation as showed in **Figure 2.2**

$$f(\xi + \alpha) = \xi' + \alpha', \qquad (2.1.11)$$

[1]Section 2.1.2 follows the derivations introduced by [7].

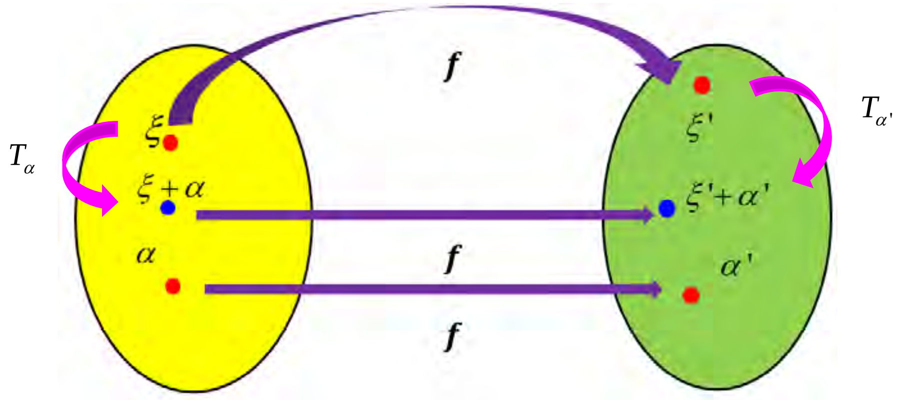

Figure 2.2: Space-time transformation between S and S' satisfying the homogeneity of space-time

where $\alpha=(\alpha_x,\alpha_t)$, $\alpha'=(\alpha_{x'},\alpha_{t'})$, and α' depends on α and f but not on ξ.
By taking $\xi=0$, from (2.1.11), one obtains

$$f(\alpha)=f(0)+\alpha'. \tag{2.1.12}$$

Substitute (2.1.11) with (2.1.12),

$$f(\xi+\alpha)=f(\xi)+f(\alpha)-f(0). \tag{2.1.13}$$

Subtracting $f(0)$ from both sides and setting $g(\xi)=f(\xi)-f(0)$, we have[2]

$$g(\xi+\alpha)=g(\xi)+g(\alpha). \tag{2.1.14}$$

Assume that g is continuous at the origin, we can write the relation

$$g(k\xi)=kg(\xi), \tag{2.1.15}$$

where k is a real number.

The additivity property in (2.1.14) and the homogeneity property in (2.1.15) imply that g is linear. Since $f(0)=0$, then $g(\xi)=f(\xi)$.
We now can write the relation between (x,t) and (x',t') in the form

$$x'=a(v)x+b(v)t,$$
$$t'=c(v)x+d(v)t. \tag{2.1.16}$$

Denoting v the velocity of the frame S relative to S'. This velocity is called the direct velocity of the pair (S,S'), which is given by

[2]The proof is given in Appendix A.
Remark: In general, $f:\mathbb{R}^{1+3}\to\mathbb{R}^{1+3}$ represents the mapping from Pseudo-Euclidean space to itself as described above. The homogeneity of spacetime and the assumption of the continuity at the origin of the function $g(\xi)=f(\xi)-f(0)$ for $0,\xi\in\mathbb{R}^{1+3}$ imply the affine or Poincare' transformations.

$$v = -\frac{b(v)}{a(v)}. \tag{2.1.17}$$

For simplicity, we can choose to consider the case when the spatial axes of the two observers have the same orientation and their times flow in the same direction, which implies the relations

$$\left.\begin{aligned} \frac{\partial x'}{\partial x} &= a(v) > 0, \\ \frac{\partial t'}{\partial t} &= d(v) > 0, \\ \frac{\partial x}{\partial x'} &= \frac{d(v)}{a(v)[d(v)+vc(v)]} > 0, \\ \frac{\partial t}{\partial t'} &= \frac{1}{d(v)+vc(v)} > 0. \end{aligned}\right\} \tag{2.1.18}$$

Define the reciprocal velocity $$w = \frac{b(v)}{d(v)} \equiv \varphi(v). \tag{2.1.19}$$

It is the velocity of S with respect to S'. We next will show that the principle of relativity together with isotropy of space and some further specified continuity assumptions, are sufficient to imply the relation

$$\phi(v) = -v. \tag{2.1.20}$$

The principle of relativity implies the existence of the set Γ of the allowed velocities of S' relative to S, which is independent of S, and that the reciprocal velocity is the same function of the direct velocity for all pairs of inertial frames. Thus, by (2.1.19), we can write

$$v = \varphi(w), \text{ or } v = \phi(\phi(v)). \tag{2.1.21}$$

Since $w \in \Gamma$, it's obvious from (2.1.21) that the codomain of the function φ is the same as its domain Γ. In addition, if $\varphi(a) = \varphi(b)$, from (2.1.21), $a = \varphi(\varphi(a)) = \varphi(\varphi(b)) = b$. This means that φ is a bijective function on $v \in \Gamma$.

Note that the condition in equation (2.1.21) is the only condition imposed by the principle of relativity on the function φ.This condition is strongly restrictive on the possible forms of φ, but without other physical assumptions about space-time, it does not imply the relation (2.1.20). For instance, $w = \varphi(v) = -\frac{v}{1-(v/c)}$, which satisfies the equation (2.1.21) without having the form of (2.1.20).

In tensor notation with Einstein Convention, $x^{\mu'} = \Lambda^{\mu'}_{\mu} x^{\mu} + a^{\mu'}$ when $\Lambda^{\mu'}_{\mu}$ denotes the Lorentz tensor. The Lorentz transformations are the special case of the Poincare' transformations, i.e., the origin is preserved after coordinates transformations or $\forall_{\mu'} a^{\mu'} = 0$. We will consider only the linear LT throughout the report.

We aim to determine the form of $\varphi(v)$ that satisfies the relation (2.1.20) by considering the isotropy of space, and space-time transformation. For the case of one spatial dimension (x-direction), isotropy of space means that no orientation along the x -axis should be considered in preference to the other.

If two frames S and S' are related by a transformation in (2.1.16), then two frames $\tilde{S}$ and $\tilde{S}'$ are obtained from the preceding ones by inverting the direction of x-axis are related by the same transformation. Therefore,

$$\begin{aligned} \tilde{x}' &= a(\tilde{v})\tilde{x} + b(\tilde{v})\tilde{t}, \\ \tilde{t}' &= c(\tilde{v})\tilde{x} + d(\tilde{v})\tilde{t}, \end{aligned} \tag{2.1.22}$$

where $\tilde{v}$ is the velocity of $\tilde{S}'$ relative to $\tilde{S}$. And $\tilde{x}' = -x'$, $\tilde{t}' = t'$, $\tilde{x} = -x$ and $\tilde{t} = t$, so that

$$\begin{aligned} \tilde{x}' &= a(v)\tilde{x} - b(v)\tilde{t}, \\ \tilde{t}' &= -c(v)\tilde{x} + d(v)\tilde{t}. \end{aligned} \tag{2.1.23}$$

Implies that $\tilde{v} = \dfrac{b(v)}{a(v)} = -v.$ Thus, Γ is symmetric, i.e., if $v \in \Gamma$, then $-v$ is so. By comparison to (2.1.23), we have the relations

$$\begin{aligned} a(-v) &= a(v), \\ b(-v) &= -b(v), \\ c(-v) &= -c(v), \\ d(-v) &= d(v). \end{aligned} \tag{2.1.24}$$

From equation (2.1.19), we arrive at the form $\varphi(-v) = -\varphi(v)$, i.e., φ is an odd function of v.

We assume the following additional assumptions to derive the reciprocity principle. These are:

(1.) The domain Γ of the function φ is an interval[3] of the real line.

(2.) φ is continuous on its domain Γ.

Since φ is a continuous bijective function and Γ is connected, then, we can state that φ is either a strictly monotonic increasing or a strictly monotonic decreasing function of v.

First, suppose that φ is strictly monotonic increasing. Let $v \in \Gamma$; then $w = \varphi(v) \in \Gamma$. Assume that $v < w$ then $\varphi(v) < \varphi(w)$ and, by (2.1.21), $w < v$, which is a contradiction. One can conclude the same way for the case of $v > w$. That is,

$$\varphi(v) = v. \tag{2.1.25}$$

[3]The definition is given in Appendix C.

If φ is supposed to be strictly monotonic decreasing, one can set $\varphi = -\varphi$. Now φ is strictly increasing, by (2.1.21) and (2.1.24), satisfies $\varphi(\varphi(v)) = v$, which is equation (2.1.20). The consistent transformation formulas of equation (2.1.25) can be written as

$$\begin{aligned} x' &= a(v)x - va(v)t, \\ t' &= c(v)x - a(v)t. \end{aligned} \qquad (2.1.26)$$

While the choice of equation (2.1.20) leads to

$$\begin{aligned} x' &= a(v)x - va(v)t, \\ t' &= c(v)x + a(v)t. \end{aligned} \qquad (2.1.27)$$

One can notice that the formulas (2.1.26) are incompatible with (2.1.18). Thus, for two observers whose space axes have the same orientation and whose times flow in the same direction, the condition (2.1.20) must necessarily hold and the corresponding transformation formulas are given by equation (2.1.27).

Formulas (2.1.27) contain the two undetermined functions of $a(v)$ and $c(v)$. The following steps will show that $c(v)$ can be expressed in terms of v and $a(v)$. Consider the inverse transformations

$$\begin{aligned} x &= \Delta^{-1}(v)a(v)x' + \Delta^{-1}(v)va(v)t', \\ t &= -\Delta^{-1}(v)c(v)x' + \Delta^{-1}(v)a(v)t', \end{aligned} \qquad (2.1.28)$$

where $\Delta(v) = a(v)[d(v) + vc(v)]$, since we confine ourselves to impose the reciprocity principle, that is, $a(v) = d(v)$. Thus,

$$\Delta(v) = a(v)[a(v) + vc(v)]. \qquad (2.1.29)$$

By the reciprocity relation, (2.1.28) can equivalently be written in the form

$$\begin{aligned} x &= a(-v)x' + va(-v)t', \\ t &= c(-v)x' + a(-v)t'. \end{aligned} \qquad (2.1.30)$$

By using (2.1.24),

$$a(v) = \frac{1}{a(v) + vc(v)}.$$

So that

$$c(v) = (1/v)(a^{-1}(v) - a(v)).$$

According to the transformation formulas (2.1.27),

$$\begin{aligned} x' &= a(v)x - va(v)t, \\ t' &= (1/v)[a^{-1}(v) - a(v)]x + a(v)t. \end{aligned} \tag{2.1.31}$$

We can use the equation (2.1.25), instead of (2.1.20), to deduce the relations (2.1.26).

From (2.1.25), $\varphi(v) = v$, implies that $a(v) = -d(v)$. So, the transformation formulas are

$$\begin{aligned} x' &= a(v)x - va(v)t, \\ t' &= -(1/v)\{[a^{-1}(v) - a(v)]x + a(v)t\}. \end{aligned} \tag{2.1.32}$$

Therefore, one can obtain the relations (2.1.26) by an inversion of the time of S'.

The transformation formulas, which connect S to an observer obtained from S' by inverting the orientation of the spatial axis, are

$$\begin{aligned} x' &= -\{a(v)x - va(v)t\}, \\ t' &= (1/v)[a^{-1}(v) - a(v)]x + a(v)t. \end{aligned} \tag{2.1.33}$$

And if S' is subjected to both a spatial and a time axes inversion, then

$$\begin{aligned} x' &= -\{a(v)x - va(v)t\}, \\ t' &= -(1/v)\{[a^{-1}(v) - a(v)]x + a(v)t\}. \end{aligned} \tag{2.1.34}$$

We note that, the essence of the principle of relativity provides that the set of all transformations in (2.1.31) - (2.1.34), as varies in Γ, forms a group $\mathcal{L}$. From this property, one can derive the explicit form of $a(v)$. Indeed, we can compose two transformations of the form in equation (2.1.31), such that

$$\begin{aligned} &\begin{pmatrix} a(v) & -va(v) \\ (1/v)(a^{-1}(v) - a(v)) & a(v) \end{pmatrix} \begin{pmatrix} a(v') & -v'a(v') \\ (1/v')(a^{-1}(v') - a(v')) & a(v') \end{pmatrix} \\ &= \begin{pmatrix} a(v)a(v') - (v/v')a(v)(a^{-1}(v') - a(v')) & -(v+v')a(v)a(v') \\ (a(v')/v)(a^{-1}(v) - a(v)) + (a(v)/v')(a^{-1}(v') - a(v')) & a(v)a(v') - (v'/v)a(v')(a^{-1}(v) - a(v)) \end{pmatrix} \\ &\equiv A(v,v'). \end{aligned} \tag{2.1.35}$$

The resulting transformation must be one of the four types (2.1.31) - (2.1.34) for some relative velocity v''. Since the determinant of transformations (2.1.31) and (2.1.34) is (+1), whereas the determinant of transformations (2.1.32) and (2.1.33) is (-1), then we will probably neglect the transformation (2.1.32) and (2.1.33).

Observe that the main diagonal elements of the matrices of both transformations, (2.1.31) and (2.1.32), are equal. That is

$$a(v)a(v') - (v/v')a(v)(a^{-1}(v') - a(v')) = a(v)a(v') - (v'/v)a(v')(a^{-1}(v) - a(v)),$$

which is $$(1/v^2)[1 - a^{-2}(v)] = (1/v'^2)[1 - a^{-2}(v')]. \quad (2.1.36)$$

Now we can write $$(1/v^2)[1 - a^{-2}(v)] \equiv K, \quad (2.1.37)$$

where K is a universal constant with the dimension of an inverse-square speed. Since $a(v)$ is positive, from the assumption (2.1.18), so that

$$a(v) = 1/\sqrt{1 - Kv^2}. \quad (2.1.38)$$

The composite velocity v'' is the negative ratio between the second and the first element of the matrix $A(v, v')$:

$$v'' = (v + v')/[1 - (v/v')\{a^{-2}(v') - 1\}] = (v + v')/(1 + Kvv'). \quad (2.1.39)$$

Three cases for different values of K are considered:

(1.) $K > 0$. One can set $c = 1/\sqrt{K}$, and formulas (2.1.31) become

$$\begin{aligned} x' &= \sqrt{1 - (v/c)^2} \cdot x - \sqrt{1 - (v/c)^2} \cdot vt, \\ t' &= -\sqrt{1 - (v/c)^2} \cdot (x \cdot v/c^2) + \sqrt{1 - (v/c)^2} \cdot t. \end{aligned} \quad (2.1.40)$$

In this case, $v \in \Gamma \equiv (-c, c)$. This is the ordinary proper orthochronous Lorentz transformations.

(2.) $K = 0$. The formulas (2.1.31) reduce to

$$\begin{aligned} x' &= x - vt, \\ t' &= t. \end{aligned} \quad (2.1.41)$$

And v varies in the domain $\Gamma \equiv (-\infty, \infty)$, which are the Galilean Transformations.

(3.) $K < 0$. Set $c = 1/\sqrt{-K}$ and formulas (2.1.31) become

$$\begin{aligned} x' &= \sqrt{1 + (v/c)^2} \cdot x - \sqrt{1 + (v/c)^2} \cdot vt, \\ t' &= \sqrt{1 + (v/c)^2} \cdot (x \cdot v/c^2) + \sqrt{1 + (v/c)^2} \cdot t. \end{aligned} \quad (2.1.42)$$

And v varies in the domain $\Gamma \equiv (-\infty, \infty)$.

Set $x^1 = x$, $x^0 = ct$, and $\tan\theta = v/c$. For $-\pi/2 < \theta < \pi/2$. From (2.1.42), we obtain

$$\begin{aligned} x'^1 &= (\cos\theta)x^1 - (\sin\theta)x^0, \\ x'^0 &= (\sin\theta)x^1 + (\cos\theta)x^0. \end{aligned} \tag{2.1.43}$$

Thus, these transformations are ordinary circular rotations $x-t$ plane, whereas the Lorentz Transformations (2.1.40) are hyperbolic rotations in $x-t$ plane. Since θ is between to $-\pi/2$. So they do not form a group. Now, if we let θ belongs to $(-\pi/2, 3\pi/2)$, a full group can be constructed. We now introduce the corresponding transformations

$$\begin{aligned} x' &= -[\sqrt{1+(v/c)^2}\cdot x - \sqrt{1+(v/c)^2}\cdot vt], \\ t' &= -[\sqrt{1+(v/c)^2}\cdot (x\cdot v/c^2) + \sqrt{1+(v/c)^2}\cdot t]. \end{aligned} \tag{2.1.44}$$

Which can be obtained from equation (2.1.42) by inverting both the spatial and time axes of S'. The rotation group (2.1.43) mathematically reflects that both space and time are isotropic.

2.2 Extended Special Relativity Formulations

Einstein's special relativity theory is based on two physical assumptions which lead to the relative velocity less than the speed of light, denoted by c, for any pairs of two inertial frames of reference. The well-known formulae for the mass, i.e., relativistic/dynamic mass, and energy are

$$m = \frac{m_0}{\sqrt{1-(u/c)^2}}, \qquad E = mc^2. \tag{2.2.1}$$

Where m_0 denotes the rest mass, obviously see that the mass formula is valid within the range of velocity $(-c, c)$, and exhibit singularity at $u = c$. The extended Lorentz transformations can be formulated by different ways.

By assuming the validity of the principle of stationary action, the Lagrangian associated with the space-like particle can be obtained. And by defining the metric corresponding to the mass, energy and momentum under the assumption that the actual physical mass is invariant as measured by any inertial observers, i.e., the energy-momentum relation. Then one can determine the corresponding mass, momentum, and energy equations, namely,

$$m = \frac{p_\infty/c}{\sqrt{(u/c)^2-1}}, \qquad E = mc^2. \tag{2.2.2}$$

Where p_∞ denotes the finite value of momentum, $p = mu$, at velocity approaches infinity. One can observe that in equation (2.2.1), the rest mass is finite and momentum is zero when velocity is zero. In contrast, in equation (2.2.2), the mass (relativistic) is zero and momentum is finite when velocity approaches infinity. In the second case, there's no concept of a rest mass, since all magnitudes of observable velocities must be greater than the speed of light.

If the requirement of invariance of the energy-momentum relation is relaxed, then the possible relations can be proposed, which are

$$m = \frac{m_\infty u / c}{\sqrt{(u/c)^2 - 1}}, \qquad E = \frac{1}{2}m(c^2 + u^2) - \frac{1}{2}m_\infty c^2 \cosh^{-1}\left(\frac{u}{c}\right) + E_0. \quad (2.2.3)$$

2.2.1 Hill-Cox Transformations

In 2012, James M.Hill and Barry J.Cox proposed the derivations of the transformations corresponding to the extension of special relativity to the superluminal motion. The obtained ELT is not the brand-new formulas, i.e., they were previously suggested by other authors in different ways. However, Hill and Cox introduced ELT based on solving a system of ordinary differential equations by treating the relative velocity of two inertial frames as a parameter which varies on $\mathbb{R}$.

In order to deduce the corresponding new transformations that are valid for $c < v < \infty$, we speculate that the new transformations and the usual Lorentz transformations are derivable from the same "pseudo-velocity" equations that give rise to equation (2.1.1) with the constraint (2.1.2), the following new transformations arising from different constraints.

We now speculate that in the regime $c < v < \infty$ that the relationship arising from (2.1.8) applies, namely $uu' = c^2$, for v tending to infinity, so that the new transformations valid for $c < v < \infty,$ the required constraints might be supplied by either

$$x' = -ct, \qquad t' = -x/c, \qquad v \to \infty, \quad (2.2.4)$$

or alternatively

$$x' = ct, \qquad t' = x/c, \qquad v \to \infty. \quad (2.2.5)$$

In terms of the characteristic variables, these constraints become respectively

$$x' + ct' = -(x + ct), \qquad x' - ct' = x - ct, \qquad v \to \infty, \quad (2.2.6)$$

$$x' + ct' = x + ct, \qquad x' - ct' = ct - x, \qquad v \to \infty. \quad (2.2.7)$$

We now view x' and t' defined in (2.1) as functions of v, while keeping x and t fixed, then differentiate x' and t' with respect to v, the pseudo-velocity equations are

$$\frac{dx'}{dv}=\frac{-t}{1-(v/c)^2}, \qquad \frac{dt'}{dv}=\frac{-x/c^2}{1-(v/c)^2}. \tag{2.2.8}$$

From a rearrangement of (2.2.8), we have

$$\left[1-\left(\frac{v}{c}\right)^2\right]\frac{dx'}{dv}=-t', \quad \left[1-\left(\frac{v}{c}\right)^2\right]\frac{dt'}{dv}=\frac{x'}{c^2}, \tag{2.2.9}$$

which becomes an autonomous system if we define a new parameter ε such that

$$\frac{d}{d\varepsilon}=\left[1-\left(\frac{v}{c}\right)^2\right]\frac{d}{dv}.$$

That is

$$\frac{dv}{d\varepsilon}=1-\left(\frac{v}{c}\right)^2.$$

Suppose we substitute $v=c\sin\phi$, this equation becomes

$$\frac{d\phi}{\cos\phi}=\frac{d\varepsilon}{c}.$$

By integration, one obtains

$$\frac{1}{2}\ln\left|\frac{1+\sin\phi}{1-\sin\phi}\right|=\frac{\varepsilon}{c}+cons\tan t.$$

Suppose we assign the value $v=v_0$ at $\varepsilon=0$, then from this equation we may deduce

$$\frac{1+v/c}{1-v/c}=\left(\frac{1+v_0/c}{1-v_0/c}\right)e^{2\varepsilon/c}. \tag{2.2.10}$$

Observe that the special relativity theories arise from the two values of v_0, $v_0=0$ corresponding to the constraint (2.1.2) and $v_0\to\infty$, which is associated with the constraints (2.2.6) or (2.2.7). From (2.2.10), we may deduce

$$v=c\tanh\left(\frac{\varepsilon}{c}\right), \qquad \text{for } 0\le v<c \tag{2.2.11}$$

and

$$v = c\coth\left(\frac{\varepsilon}{c}\right), \qquad \text{for } c < v < \infty. \tag{2.2.12}$$

Observe that if we treat the parameter ε in these two equations as being the same, then they suggest a connection that the new theory reflects the Einstein theory with the velocity replaced by c^2/v. In either case, the two differential equations (2.2.9) become

$$\frac{dx'}{d\varepsilon} = -t', \qquad \frac{dt'}{d\varepsilon} = -\frac{x}{c^2}.$$

One may differentiate either equation with respect to ε, in this case we deduce

$$x'(\varepsilon) = A\sinh\left(\frac{\varepsilon}{c}\right) + B\cosh\left(\frac{\varepsilon}{c}\right), \tag{2.2.13}$$

and

$$t'(\varepsilon) = -\frac{[A\cosh(\varepsilon/c) + B\sinh(\varepsilon/c)]}{c}. \tag{2.2.14}$$

Where A and B denote arbitrary constants of integration. In the case of the usual Lorentz transformations, we have from (2.1.2), that $A = -ct$ and $B = x$. That is

$$x'(\varepsilon) = x\cosh\left(\frac{\varepsilon}{c}\right) - ct\sinh\left(\frac{\varepsilon}{c}\right), \tag{2.2.15}$$

and

$$t'(\varepsilon) = t\cosh\left(\frac{\varepsilon}{c}\right) - \frac{x}{c}\sinh\left(\frac{\varepsilon}{c}\right). \tag{2.2.16}$$

By using the equation (2.2.11) together with the hyperbolic function identity, implies that

$$\cosh\left(\frac{\varepsilon}{c}\right) = \frac{1}{\sqrt{1-(v/c)^2}}, \qquad \sinh\left(\frac{\varepsilon}{c}\right) = \frac{v/c}{\sqrt{1-(v/c)^2}}. \tag{2.2.17}$$

For the case subjected to the constraint (2.2.4), we find $A = x$ and $B = -ct$, from (2.2.13) - (2.2.14), we have

$$x'(\varepsilon) = x\sinh\left(\frac{\varepsilon}{c}\right) - ct\cosh\left(\frac{\varepsilon}{c}\right),$$

$$t'(\varepsilon) = t\sinh\left(\frac{\varepsilon}{c}\right) - \frac{x}{c}\cosh\left(\frac{\varepsilon}{c}\right).$$

Using the equation (2.2.12) together with the hyperbolic function identity, we have

$$\sinh\left(\frac{\varepsilon}{c}\right)=\frac{1}{\sqrt{1-(v/c)^2}},\qquad \cosh\left(\frac{\varepsilon}{c}\right)=\frac{v/c}{\sqrt{1-(v/c)^2}}. \tag{2.2.18}$$

Noting that in particular that for $\varepsilon > 0$ we have adopted the positive square root for $\sinh(\varepsilon / c)$. Altogether, we find the transformations

$$x'=\frac{x-vt}{\sqrt{(v/c)^2-1}},\qquad t'=\frac{t-vx/c^2}{\sqrt{(v/c)^2-1}}. \tag{2.2.19}$$

These can be considered as the Lorentz transformations that applies for $c < v < \infty$.

The other form of transformations arises from the constraint presented in equation (2.2.5). In this case, we find that $A=-x$ and $B=ct$. We now have

$$x'=\frac{-x+vt}{\sqrt{(v/c)^2-1}},\qquad t'=\frac{-t+vx/c^2}{\sqrt{(v/c)^2-1}}. \tag{2.2.20}$$

We observe that the transformation formulas in (2.2.20) can be obtained by inverting equation (2.2.19), i.e., opposite spatial and time axes orientations.

2.2.2 Implications of Universality of Velocities Addition Formula

$$u=\frac{u'+v}{1+u'v/c^2}\equiv u'\oplus_E v. \tag{2.2.21}$$

Note that this formula is valid only in the case that the directions of u, u', and v are co-linear to one another. For more general formula, the reader can see Abraham A. Ungar, "Einstein's velocity addition law and its hyperbolic geometry" [6].

When $|u'|,|v|<c$, this leads to the result $|u|<c$, which means that superluminal velocities cannot be achieved by the addition of any two subluminal velocities. This particular case forms a commutative or abelian group under an operator " $\oplus_E$ ". One can check that as $|v|\to\infty, |u'|\to\infty$, then $uu'=c^2$ and $uv=c^2$ respectively.

When $|u'|,|v|>c$, then $|u|<c$ which is the same as the when $|u'|,|v|<c$. If either $|u'|$ or $|v|$ is greater than c Not both), then $|u|>c$. These reflect[4,5] the equivalence of subluminal set and superluminal set. For these cases, a group under $\oplus_E$ cannot be constructed.

[4]The reader can study in detail in [2].

[5]These consequences will be taken into account in the Chapter 3.

2.2.3 Review of Dynamics in Special Relativity and Extended Special Relativity

[6]First, we assume that the principle of stationary action also applies to FTL particles. This follows directly from the assumption of homogeneity of space and time because this is the shared condition for both LT and ELT. (Isotropy of space-time is distinct in the two cases), and we know that this principle is true for slower than light particles.

For a free particle[7], i.e., a particle that is not exerted by a non-zero total force or being free from any external potentials, so the particle moves along the space-time geodesic, which reduces to a straight line in the absence of gravity. This implies that the differential of action is proportional to the line element of the particle in spacetime, that is

$$dS \equiv \alpha ds, \qquad ds \equiv \sqrt{\left|c^2dt^2 - dx^2 - dy^2 - dz^2\right|}. \tag{2.2.22}$$

We note that α can take different values at different regimes, namely subluminal and superluminal regimes.

In the case of a forward, i.e., forward in time, and time-like particle, we have the form

$$dS = L(u)dt, \qquad L(u) = \alpha c\sqrt{1-(u/c)^2}. \tag{2.2.23}$$

Where $L(u)$ represents the Lagrangian as a function of particle's velocity u.
Note that in this case, the Hamiltonian is the same as the total energy, energy for short, of the system, so we can determine the expressions of the energy and the (canonical) momentum by the formulas

$$p(u) = \frac{\partial L(u)}{\partial u}, \qquad E(u) = u\left[\frac{\partial L(u)}{\partial u}\right] - L(u). \tag{2.2.24}$$

From equations (2.2.23) and (2.2.24), we have

$$p(u) = -\frac{\alpha u/c}{\sqrt{1-(u/c)^2}}, \qquad E(u) = -\frac{\alpha c}{\sqrt{1-(u/c)^2}}. \tag{2.2.25}$$

To determine α, we may use the fact that for $u < c$, the Galilean transformation is dominant. If we expand the momentum in a power series of u/c and keep only the first (linear) order of u/c. We thus obtain $p \approx -\alpha u/c$, while the Newtonian mechanics defines momentum as $p = mu$, that is $\alpha = -mc$. Now we can write the complete form of (2.2.25) as

$$p(u) = \frac{mu}{\sqrt{1-(u/c)^2}}, \qquad E(u) = \frac{mc^2}{\sqrt{1-(u/c)^2}}. \tag{2.2.26}$$

[6]Section 2.2.3 is based on [1] and [2].
[7]A particle in this context refers to a "point particle", which we neglect its shape and the effect of spacetime curvature due to its mass.

The relations in equation (2.2.26) are the usual expressions in special relativity.

In the case of forward space-like particle, the Lagrangian takes the form

$$L(u) = \alpha c\sqrt{(u/c)^2 - 1}. \quad (2.2.27)$$

Follows from (2.2.24), we obtain

$$p(u) = \frac{\alpha u/c}{\sqrt{(u/c)^2 - 1}}, \quad E(u) = \frac{\alpha c}{\sqrt{(u/c)^2 - 1}}. \quad (2.2.28)$$

Since, these expressions cannot be reduced to the known special case, e.g., Galilean Transformations, so we need to determine α by purely mathematical approach together with some additional physical assumptions.
Observe the limit as $u \to \pm\infty$, which gives us

$$\lim_{u\to\infty} p(u) = \alpha, \qquad \lim_{u\to\infty} E(u) = 0. \quad (2.2.29)$$

From the equation (2.2.29), we observe that α is the momentum at the infinite speed, which can be denoted by p_∞.

Since the (actual) mass[8] of a particle must be a universal invariant, then it is possible to define a metric in the space of energy and momenta as

$$c^2 d\tilde{m} = \sqrt{\left|dE^2 - c^2 dp_x^2 - c^2 dp_y^2 - c^2 dp_z^2\right|}. \quad (2.2.30)$$

This relation is called "energy-momentum relation", which is invariant.

Thus, it follows that the mass, the energy and the momentum must always be related by the formula $\tilde{m}c^2 = \sqrt{E^2 - c^2 p^2}$. If we put $E = 0$ in this last equation, we get $p = \tilde{m}c$ (we have assumed that p is positive for u is positive). Now we can determine that $\alpha = p_\infty = \tilde{m}c$, so the expressions of momentum and energy are

$$p(u) = \frac{\tilde{m}u}{\sqrt{(u/c)^2 - 1}}, \qquad E(u) = \frac{\tilde{m}c^2}{\sqrt{(u/c)^2 - 1}}. \quad (2.2.31)$$

Alternatively, the dynamics relations corresponding to both special relativity and extended special relativity can be derived as follows[9]

[8]Mass refers to a physical quantity which connects energy and momentum in the form of a metric in energy-momentum space. This mass is not the same as the relativistic mass.
[9]The derivations of mass and energy relations are introduced by Hill-Cox in [2].

Using the relation $E = mc^2$, Einstein derived the mass formula $m = \dfrac{m_0}{\sqrt{1-(u/c)^2}}$ from the energy-momentum equations

$$\frac{dE}{dt} = Fu, \qquad F = \frac{d}{dt}(mu). \tag{2.2.32}$$

Using energy-mass relation, one obtains

$$\frac{dE}{dt} - u\frac{d}{dt}(mu) = -\frac{c^2}{2m}\frac{d}{dt}\left\{m^2\left[\left(\frac{u}{c}\right)^2 - 1\right]\right\}.$$

If we consider $0 \le |u| < c$, the above equation implies that $m\left[1-(u/c)^2\right]^{1/2} = constant \equiv m_0$, which is the mass formula arises from special relativity. Notice that m_0 denotes the value of (relativistic) mass m when u is zero, which is called the "rest mass" and it is the only actual mass of a particle.

In the case that $0 \le |v| < c$, we deduce from $p = mu$ and $p' = m'u'$, and by the velocities addition formula, when v represents the relative velocity of S' relative to S, the Lorentz invariant energy-momentum relations

$$p' = \frac{p - vm}{\sqrt{1-(v/c)^2}}, \qquad m' = \frac{m - vp/c^2}{\sqrt{1-(v/c)^2}}, \tag{2.2.33}$$

which give

$$p'^2 - (cm')^2 = p^2 - (cm)^2. \tag{2.2.34}$$

Assume that that for the case of $c < |u| < \infty$, the formula $E = mc^2$ is still valid. We may deduce the mass formula by using energy-momentum equations (2.2.32) as the previous case, that is

$$\frac{dE}{dt} - u\frac{d}{dt}(mu) = -\frac{c^2}{2m}\frac{d}{dt}\left\{m^2\left[\left(\frac{u}{c}\right)^2 - 1\right]\right\}.$$

The above formula implies the other form of mass formula, namely

$$m\left[(u/c)^2 - 1\right]^{1/2} = constant \equiv \widetilde{m},$$

which is valid only for $c < u < \infty$. In this case, as stated previously, $\widetilde{m}$ denotes the value of (relativistic) mass m when $u = \sqrt{2}c$, which ***cannot*** be called the "rest mass".

If we consider the case that $c < |v| < \infty$, and assume to write $p = mu$ and $p' = m'u'$, together with the velocities addition formula, where v represents the relative velocity of S' relative to S. The ***extended Lorentz*** invariant energy-momentum relations are obtained, which are

$$p' = \frac{p - vm}{\sqrt{(v/c)^2 - 1}}, \qquad m' = \frac{m - vp/c^2}{\sqrt{(v/c)^2 - 1}}. \tag{2.2.35}$$

So that

$$p'^2 - (cm')^2 = (cm)^2 - p^2. \tag{2.2.36}$$

Recall the velocities addition formula (2.2.21) $u = \dfrac{u' + v}{1 + u'v/c^2} \equiv u' \oplus_E v.$

Since $c < |v| < \infty$, so if $c < |u'| < \infty$, then $0 \le |u| < c$. One can deduce from equation (2.2.36) the relation

$$m'\left[\left(\frac{u'}{c}\right)^2 - 1\right]^{1/2} = m\left[1 - \left(\frac{u}{c}\right)^2\right]^{1/2} = cons\tan t. \tag{2.2.37}$$

In particular, if the relative speed between two inertial frames is superluminal or FTL, then either observer is possible to measure m_0 whereas the other is possible to measure $\widetilde{m}$. If we insist on the invariance of mass, we can interpret that the "actual mass" of a particle is the "rest mass", m_0, which is the same as $\widetilde{m}$.

However, if we do not insist on the energy-momentum relations (2.2.35) and we would like to propose a mass variation expression[10] that contains a finite value m_∞ when velocity approaches infinity. By the relation $E = mc^2$, together with suggestive nature of the LT and ELT, it is reasonable to propose the relation (2.2.3).

Thus, if we do not impose the invariance requirement, the corresponding energy calculation is as follows. We assume that the energy $E = mc^2\phi(\xi)$ for some function $\phi(\xi)$, where $\xi = (u/c)^2$. In terms of ξ, the energy and momentum balance, equation (2.2.32), become

$$\frac{d}{d\xi}\left[\left(\frac{\xi}{\xi - 1}\right)^{1/2} \varphi(\xi)\right] = \xi^{1/2} \frac{d}{d\xi}\left(\frac{\xi}{(\xi - 1)^{1/2}}\right).$$

The above ordinary differential equation can be rearranged to yield

[10]A mass variation expression is suggested by Hill-Cox in [2].

$$\frac{d}{d\xi}\left[\left(\frac{\xi}{\xi-1}\right)^{1/2}(\varphi(\xi)-1)\right]=\frac{1}{2}\left(\frac{\xi-1}{\xi}\right)^{1/2}.$$

By integration gives

$$\phi(\xi)=\frac{1}{2}(1+\xi)-\frac{1}{2}\left(\frac{\xi-1}{\xi}\right)^{1/2}\cosh^{-1}(\xi^{1/2})+\phi_0\left(\frac{\xi-1}{\xi}\right)^{1/2}.$$

Where ϕ_0 denotes the arbitrary constant of integration. In terms of the original variables, we may finally deduce

$$E=\frac{m}{2}(c^2+u^2)-\frac{m_\infty c^2}{2}\cosh^{-1}\left(\frac{u}{c}\right)+\varphi_0 m_\infty c^2. \tag{2.2.38}$$

This equation reflects that ϕ_0 can be determine by the zero-energy condition.

And the corresponding mass equation is $m=\frac{m_\infty u/c}{\sqrt{(u/c)^2-1}}$, which is stated in equation (2.2.3).

Finally, for any mass variation $m(u)$, the energy-momentum equations (2.2.32) admit the general integral form, for $c<|u|<\infty$,

$$E(u)+\int_c^u wm(w)dw=m(u)u^2+cons\tan t. \tag{2.2.39}$$

By writing equation (2.2.32) as $\frac{dE}{du}=u\frac{d}{du}(mu)$, and by addition of mu on both sides, then we obtain the integral (2.2.39). We may use equation (2.2.39) to deduce the energy expression associated with any mass variation $m(u)$. For example, for $c<|u|<\infty$, and $m(u)$ is defined by

$$m(u)=\frac{m_0+m_\infty u/c}{\sqrt{(u/c)^2-1}}.$$

The associated form of energy is

$$E=mc^2+\frac{m_\infty c^2}{2}\left\{\left(\frac{u}{c}\right)\left[\left(\frac{u}{c}\right)^2-1\right]^{1/2}-\cosh^{-1}\left(\frac{u}{c}\right)\right\}+E_0,$$

including both the formula deduced from equation (2.3.32) when m_∞ is zero, and the formula in equation (2.82) when m_0 is zero.

2.3. Inconsistency of Hill-Cox Transformations with Special Relativity

The main issue of Hill-Cox transformations (Including the further derived ELT, which are the modification of the Hill-Cox transformations) is that it is inconsistent to special relativity in spacetime with higher than 1 spatial dimension, i.e., the usual (general) spacetime in special relativity is the four-dimensional (1,3)-hyperbolic structure[11]. The stated earlier, the extended special relativity described by the ELT, which leads to the invalidity of the principle of invariance of the speed of light, except in two-dimensional spacetime.

Define

$$HC1: t' = \frac{-t + vx/c^2}{\sqrt{(v/c)^2 - 1}}, \qquad x' = \frac{-x + vt}{\sqrt{(v/c)^2 - 1}}, \qquad y' = y, \qquad z' = z. \tag{2.3.1}$$

$$HC2: t' = \frac{t - vx/c^2}{\sqrt{(v/c)^2 - 1}}, \qquad x' = \frac{x - vt}{\sqrt{(v/c)^2 - 1}}, \qquad y' = y, \qquad z' = z. \tag{2.3.2}$$

where v is the superluminal relative speed of the two inertial observers.

To see what the Hill-Cox transformations do with the light cones if dimension $d > 2,$ first we show that they can be written as the compositions of the Lorentz transformations and a transformation exchanging the time axis and a spatial axis, if we use relativistic units (i.e., set the speed of light c to be 1), and a scaling of time is included in the below visualization as well.

Let v, c be as in (2.84) and (2.85). Let L be the Lorentz boost in relativistic units, corresponding to velocity c/v, i.e., L maps (t, x, y, z) to (t', x', y', z'), where

$$L: t' = \frac{t - (c/v)x}{\sqrt{1 - (c/v)^2}}, \qquad x' = \frac{x - (c/v)t}{\sqrt{1 - (c/v)^2}}, \qquad y' = y, \qquad z' = z. \tag{2.3.3}$$

Note that $c/v < 1$ if $v > c$. Let σ_1 and σ_2 be the two transformations as follows, $\sigma_1 : (t, x, y, z) \mapsto (x, t, y, z)$, $\sigma_2 : (t, x, y, z) \mapsto (-x, -t, y, z),$ and finally let ρ be a transformation that scales the time coordinate with c , i.e., $\rho : (t, x, y, z) \mapsto (ct, x, y, z)$. Then one can show that

$$HC_i = \rho^{-1} \circ \sigma_i \circ L \circ \rho. \tag{2.3.4}$$

[11]The convention of spacetime coordinates in this section is $(t, x, y, z).$

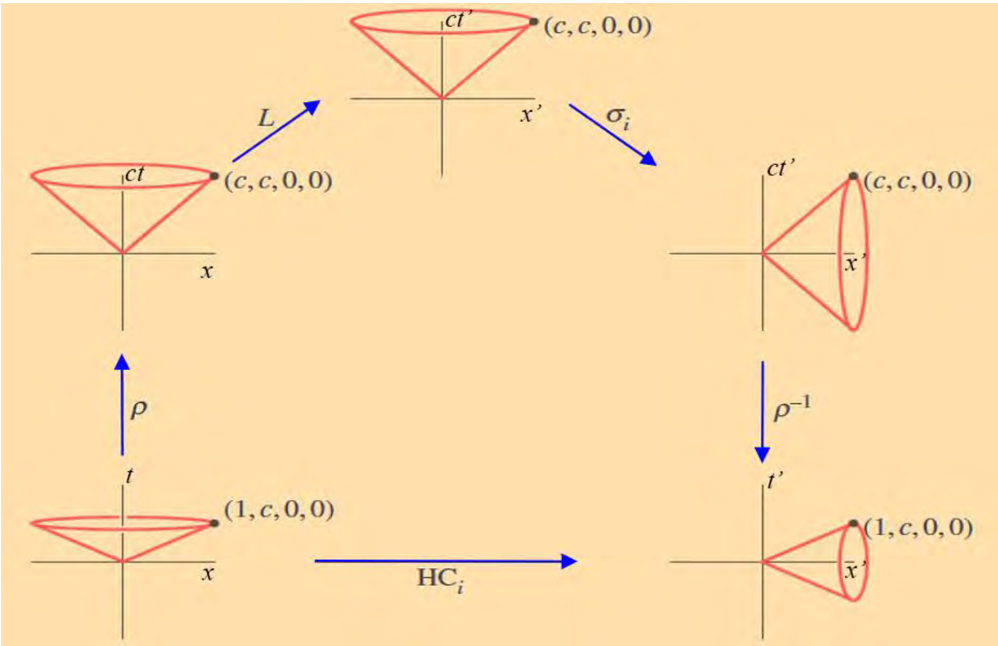

Figure 2.3: Decomposition of Hill-Cox transformations
(Modification is based on Andreka *et al*, Proc. R. Soc. A, 469, 2154 (2013))

The decomposition in **Figure 2.3** shows what HC_i affect light cones. The key point will be that the transformations σ_1 and σ_2 radically deform the light cones in $d > 2$, namely they only preserve the light cones in two-dimensional spacetime.

$$(\rho^{-1} \circ \sigma_1 \circ L \circ \rho)(t, x, y, z) = (\rho^{-1} \circ \sigma_1 \circ L)(ct, x, y, z)$$

$$= (\rho^{-1} \circ \sigma_1)\left(\frac{ct-(c/v)x}{\sqrt{1-(c/v)^2}}, \frac{x-(c/v)ct}{\sqrt{1-(c/v)^2}}, y, z \right)$$

$$= \rho^{-1}\left(\frac{x-(c/v)ct}{\sqrt{1-(c/v)^2}}, \frac{ct-(c/v)x}{\sqrt{1-(c/v)^2}}, y, z \right)$$

$$= \left(\frac{x/c-(c/v)t}{\sqrt{1-(c/v)^2}}, \frac{ct-(c/v)x}{\sqrt{1-(c/v)^2}}, y, z \right)$$

$$= \left(\frac{(c/v)(vx/c^2-t}{(c/v)\sqrt{(v/c)^2-1}}, \frac{(c/v)(vt-x)}{(c/v)\sqrt{(v/c)^2-1}}, y, z \right)$$

$$= \left(\frac{-t + vx/c^2}{\sqrt{(v/c)^2 - 1}}, \frac{-x + vt}{\sqrt{(v/c)^2 - 1}}, y, z \right)$$

$$= HC_1(t, x, y, z).$$

The composition for HC_2 is similar, i.e., by replacing σ_1 with σ_2. The obtained result is the transformations, which are composed of opposite spatial and time orientations compared to HC_1.

Consider the equation of the light cone (whose apex is the origin) in the (t, x, y, z) coordinate system is

$$(ct)^2 = x^2 + y^2 + z^2. \tag{2.3.5}$$

We obtain the HC_i-image of this by successively applying the operators, ρ, L, σ_i and ρ^{-1} to this equation, and we get both cases of i =1,2

$$x^2 = (ct)^2 + y^2 + z^2. \tag{2.3.6}$$

Equation (2.3.6) is related to the flipped-over light cone presented in the bottom right corner of the figure 3. Let l be any line going through the origin and orthogonal to the time axis (see **Figure 2.4**).

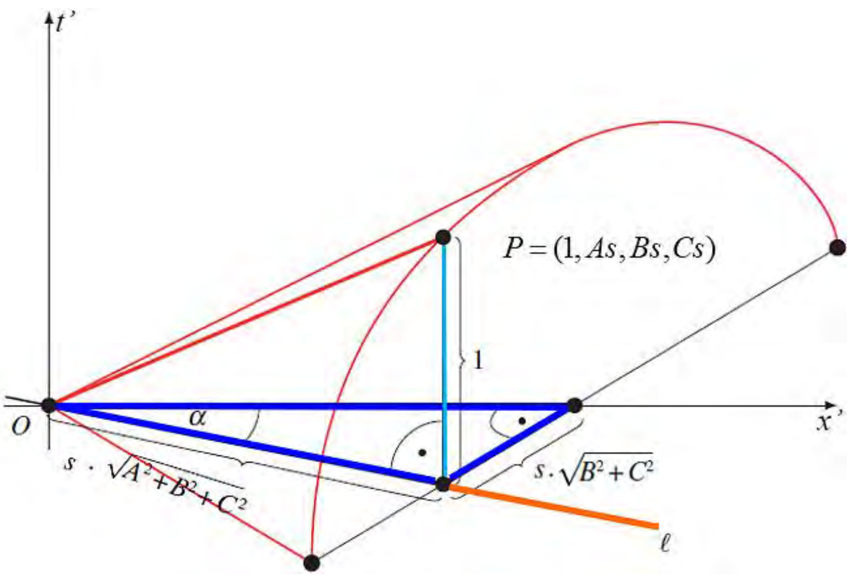

Figure 2.4: Illustration for the relative speed of light that depends on spatial directions as observed in the frame, which the light cone equation is (2.3.6).
(Adapted from Andreka *et al*, Proc. R. Soc. A, 469, 2154 (2013))

Then, there are A, B, C such that l's equation is

$$(0, As, Bs, Cs), \qquad s \in \mathbb{R}. \tag{2.3.7}$$

Thus, the point p of the light cone in direction l and with time coordinate 1 is

$$(1, As, Bs, Cs), \quad \text{with } (As)^2 = c^2 + (Bs)^2 + (Cs)^2, \tag{2.3.8}$$

so, we obtain

$$s = \frac{c}{\sqrt{A^2 - B^2 - C^2}}. \tag{2.3.9}$$

Thus, the relative speed of light in the direction l is

$$c(l) = \frac{c\sqrt{A^2 + B^2 + C^2}}{\sqrt{A^2 - B^2 - C^2}}. \tag{2.3.10}$$

Let α denotes the angle between l and the $x-$axis, then $\tan(\alpha) = \dfrac{\sqrt{B^2 + C^2}}{A}$. Substituting this to (2.3.10), we obtain

$$c(\alpha) = c \cdot \sqrt{\frac{1 + \tan^2(\alpha)}{1 - \tan^2(\alpha)}}, \qquad \text{for } 0 \leq \alpha < 45^\circ. \tag{2.3.11}$$

The equation (2.3.11) shows that $c(\alpha)$ is the speed of light c when α is zero, i.e., light travels along the $x-$axis. Thus, the observer who is stationary in this inertial frame cannot send out the light signal in any spatial directions, except the $x-$direction. Due to this fact, the principle of invariance of the speed of light is not valid anymore.

Chapter 3

Results and Discussion

First, we will give the derivation of the ELT based on the reciprocity principle, which is more transplant than Hill-Cox derivations, about the anisotropy of space that affects the space-time transformations of two inertial frames. We further modify the Hill-Cox derivations to obtain the previous transformations. We can equivalently consider how the two types of Hill-Cox transformations affect the light cones in reversed operations by using the similar methods presented in section 2.3 in which we will show the crucial result of the anisotropy of space that governs the characteristic property of these particular transformations. However, according to the following derivations, the two types of Hill-Cox transformations are equivalent, i.e., they contain the opposite spatial and time orientations, and do not contain the different physical perspectives. From this point, if one of them is applied to describe the events in spacetime, and since there is a preferred or privileged direction in one inertial frame of reference, the other transformation cannot be applied.

Moreover, we will discuss the main problem of the obtained transformations, namely, the invariance of the speed of light, which is not valid in the inertial frame that its space is not isotropic. We will also point out the remarkable result that a particular frame can know that he is travelling FTL by considering the intersection of a time-hyperplane with the light cone, the result is that the observer in this frame can observe only the propagation of light along the $x-$axis, i.e., one photon travels forward in time along the $x-$direction, and the other photon travels backward in time to the same spatial direction. This result is partly consistent with the proposed description [3]. Since we consider the condition of anisotropy of space mathematically rigorous, the transplant obtained result leads to some arguments which are not included in the previous work.

The solution of the invalidity of the invariance of the speed of light can be possibly resolved by introducing the six-dimensional spacetime with some appropriate conditions. We will review this particular solution which was suggested by the previous author [1]. We will further suggest the new structure of spacetime, called the $(n,3)$-ultra-hyperbolic structure, under the condition that only "light or photon (massless particle)" is the privilege system that can access any n-degree of freedom of time coordinates.

We also provide two important comments on this work. First, if assume the "switching principle" four-dimensional ELT. We arrive at the conclusion that the photon travels backward in time along the $x-$direction is just the anti-photon that travels along the $x-$direction.

However, the anti-photon is the same as the photon, we can conclude that the observer in this particular frame of reference should observe the two photons, which we may consider as the bound state, move along the $x-$axis for any instance of time. The last comment is due to the derivation of ELT, which restricts the domain of a function that connects the direct velocity and the corresponding reciprocal velocity must be the same as the codomain. This implies the isolated pair of inertial frames. But we somehow propose the special configurations of more than two inertial frames and by considering the mappings associated with the subluminal and superluminal sets of velocities.

3.1 Derivation of the Reciprocity Principle and the Corresponding ELT

Recall the result of homogeneity of space and time in section 2.1.2, then we can write

$$\begin{aligned} x' &= a(v)x + b(v)t, \\ t' &= c(v)x + d(v)t. \end{aligned} \tag{3.1.1}$$

The direct velocity, i.e., the relative velocity of S' with respect to S is

$$v = -\frac{b(v)}{a(v)}. \tag{3.1.2}$$

One can write the relations

$$\left.\begin{aligned} \frac{\partial x'}{\partial x} &= a(v), \\ \frac{\partial t'}{\partial t} &= d(v), \\ \frac{\partial x}{\partial x'} &= \frac{d(v)}{a(v)[d(v)+vc(v)]}, \\ \frac{\partial t}{\partial t'} &= \frac{1}{d(v)+vc(v)}. \end{aligned}\right\} \tag{3.1.3}$$

Note that, in this case, we still do not assign any sign to each equation of equation (3.1.3). Because we will further show that those coefficients deeply depend on the sign of v.
Denoting by w the reciprocal velocity, namely, the velocity of S with respect to, we have

$$w = \frac{b(v)}{d(v)} \equiv \varphi^*(v). \tag{3.1.4}$$

To derive the reciprocity principle, we need to assume that for any pairs of two inertial frames, there exists a function that maps any direct velocities to the corresponding reciprocal velocities and its inverse are itself. However, the principle of relativity implies that this condition must be the same for all inertial frames, i.e., there must be only one so-called function that is valid for all inertial frames.

Now we can write $$v = \varphi^*(w), \tag{3.1.5}$$

or

$$\varphi^*(\varphi^*(v)) = v. \tag{3.1.6}$$

From section 2.1.2, we know that $\varphi^* : \Gamma^* \to \Gamma^*$ is a bijective function on v.

Now we assume the condition that the space is not isotropic by considering the following procedures,

Suppose we invert the $x-$axis and keeping the orientation of time fixed, the corresponding frame is denoted by $\bar{S}$ with coordinates $(\bar{x},\bar{t})$. And the $\bar{S}'$ has the fixed $x'-$axis, but the orientation of time is reversed, its coordinates are $(\bar{x}',\bar{t}')$. The essential assumption is that we connect $\bar{S}$ and $\bar{S}'$ with the same transformations, which relate S and S', i.e., equation (3.1.1). We have

$$\begin{aligned} \bar{x}' &= a(\bar{v})\bar{x} + b(\bar{v})\bar{t}, \\ \bar{t}' &= c(\bar{v})\bar{x} + d(\bar{v})\bar{t}, \end{aligned} \tag{3.1.7}$$

where $\bar{v}$ is the velocity of $\bar{S}'$ relative to $\bar{S}$. On the other hand, $\bar{x} = -x$, $\bar{t} = t$, $\bar{x}' = x'$, and $\bar{t}' = -t'$. So that

$$\begin{aligned} \bar{x}' &= -a(v)\bar{x} + b(v)\bar{t}, \\ \bar{t}' &= \ \ c(v)\bar{x} - d(v)\bar{t}. \end{aligned} \tag{3.1.8}$$

we can conclude that

$$\begin{aligned} a(-v) &= -a(v), \\ b(-v) &= \ \ b(v), \\ c(-v) &= \ \ c(v), \\ d(-v) &= -d(v). \end{aligned} \tag{3.1.9}$$

And $\bar{v} = b(v)/a(v) = -v.$ Thus, $\varphi^*(-v) = -\varphi^*(v).$

We rather propose the strictly monotonic property of the function initially and together with:
(1.) The domain Γ^* of the function φ^* is the subset of the real line[12].
(2.) φ^* is continuous on its domain Γ^*.

[12]In this section, we do not assume that Γ^* is an interval on $\mathbb{R}$. Because we expect to obtain the set Γ^*, which is symmetrical due to the reciprocity principle. The detail is provided in Appendix C.

Follows the same previous steps, we arrive at two types of transformations, which are

$$\begin{aligned} x' &= a(v)x - va(v)t, \\ t' &= c(v)x - a(v)t. \end{aligned} \tag{3.1.10}$$

And

$$\begin{aligned} x' &= a(v)x - va(v)t, \\ t' &= c(v)x + a(v)t. \end{aligned} \tag{3.1.11}$$

Equation (3.1.10) corresponds to the relation $\varphi^*(v) = v$, whereas equation (3.1.11) is associated with the relation $\varphi^*(v) = -v$.

Suppose we consider the transformations (3.1.11), which the reciprocity principle emerges, i.e., $\varphi^*(v) = -v$, which implies $a(v) = d(v)$. We can determine $c(v)$ in terms of $a(v)$. Indeed, consider the inverse transformations

$$\begin{aligned} x &= \ \ \Delta^{-1}(v)a(v)x' + \Delta^{-1}(v)va(v)t', \\ t &= -\Delta^{-1}(v)c(v)x' + \Delta^{-1}(v)a(v)t', \end{aligned} \tag{3.1.12}$$

where

$$\Delta(v) = a(v)[a(v) + vc(v)]. \tag{3.1.11}$$

By the reciprocity relation, (3.1.12) can equivalently be written in the form

$$\begin{aligned} x &= a(-v)x' + va(-v)t', \\ t &= c(-v)x' + d(-v)t'. \end{aligned} \tag{3.1.14}$$

By using (3.1.9), we obtain

$$-a(v) = \frac{a(v)}{a(v)[a(v) + vc(v)]},$$

$$c(v) \ = -(1/v)(a^{-1}(v) + a(v)). \tag{3.1.15}$$

Thus,

$$\begin{aligned} x' &= a(v)x - va(v)t, \\ t' &= -(1/v)[a^{-1}(v) + a(v)]x + a(v)t. \end{aligned} \tag{3.1.16}$$

Now, if we consider the transformations in equation (3.1.10), which implies $a(v) = -d(v)$.

Note that, in this case,

$$\Delta(v) = a(v)[-a(v) + vc(v)].$$

Using equation (3.1.12), we have

$$a(v) = \frac{a(v)}{a(v)[-a(v) + vc(v)]},$$

$$c(v) = (1/v)[a^{-1}(v) + a(v)]. \tag{3.1.17}$$

Thus,

$$\begin{aligned} x' &= a(v)x - va(v)t, \\ t' &= (1/v)[a^{-1}(v) + a(v)]x - a(v)t. \end{aligned} \tag{3.1.18}$$

Obviously, equation (3.1.18) can be obtained by inverting the time direction in equation (3.1.16). Unlike the previous case in which the space is isotropic as described in section (2.1.2), in the current case, it is not allowed to invert the orientation of spatial direction (there is only one preferred direction, the revered spatial direction does not contribute any physical meaning).

To determine the form of $a(v)$, we may need to compose two transformations of either equation (3.1.16) or equation (3.1.18) themselves. Recall that the determinant of the transformations in equation (3.1.16) is -1, and the determinant of the transformations in equation (3.1.18) is +1. By composing the same two transformations, we expect to obtain the transformation with the same determinant, i.e., $\det(AB) = \det(A)\cdot\det(B)$, and in this case the two types of transformations have either +1 or -1 determinant. Therefore, we insist on the transformation in equation (3.1.18). That is

$$\begin{aligned} &\begin{pmatrix} a(v) & -va(v) \\ (1/v)(a^{-1}(v) + a(v)) & -a(v) \end{pmatrix} \begin{pmatrix} a(v') & -v'a(v') \\ (1/v')(a^{-1}(v') + a(v')) & -a(v') \end{pmatrix} \\ &= \begin{pmatrix} a(v)a(v') - (v/v')a(v)(a^{-1}(v') + a(v')) & (v - v')a(v)a(v') \\ (a(v')/v)(a^{-1}(v) + a(v)) - (a(v)/v')(a^{-1}(v') + a(v')) & a(v)a(v') - (v'/v)a(v')(a^{-1}(v) + a(v)) \end{pmatrix} \\ &\equiv A_1(v, v'). \end{aligned} \tag{3.1.19}$$

Since the main diagonal elements of the matrices of the transformations in equation (3.1.18) have an opposite sign each other. Therefore, we can write

$$\begin{aligned} a(v)a(v') - (v/v')a(v)(a^{-1}(v) + a(v)) &= (v'/v)a(v')(a^{-1}(v') + a(v')) - a(v)a(v'), \\ 2 &= v \cdot \{(1/v')(1 + a^{-2}(v')\} + v' \cdot \{(1/v)(1 + a^{-2}(v)\}. \end{aligned}$$

One can see that we cannot separate equation (3.1.19) into two terms, which one depends on v and the other depends on v'. Therefore, we cannot determine the exact form of the coefficient "a", i.e., physically, "a" should be the same form for any frames of reference.

The other less possible form, regarding its group property, is that the transformations in equation (3.1.16). Follow the same procedures, we have

$$\begin{pmatrix} a(v) & -va(v) \\ -(1/v)(a^{-1}(v) + a(v)) & a(v) \end{pmatrix} \begin{pmatrix} a(v') & -v'a(v') \\ -(1/v')(a^{-1}(v') + a(v')) & a(v') \end{pmatrix}$$
$$= \begin{pmatrix} a(v)a(v') + (v/v')a(v)(a^{-1}(v') + a(v')) & -(v+v')a(v)a(v') \\ -\{(a(v')/v)(a^{-1}(v) + a(v)) + (a(v)/v')(a^{-1}(v') + a(v'))\} & a(v)a(v') + (v'/v)a(v')(a^{-1}(v) + a(v)) \end{pmatrix}$$
$$\equiv A_2(v, v'). \tag{3.1.20}$$

Since the main diagonal elements of the matrices of the transformations are equal, we then have

$$(v/v')a(v)(a^{-1}(v) + a(v)) = (v'/v)a(v')(a^{-1}(v') + a(v')),$$

which can be written as

$$(1/v^2)\{1 + a^{-2}(v)\} = (1/v'^2)\{1 + a^{-2}(v')\} \equiv k \in \mathbb{R}.$$

Therefore,
$$a(v) = \pm \frac{1}{Kv^2 - 1}. \tag{3.1.21}$$

Note that these particular forms of transformations are improper due to the closure property. However, we propose an additional assumption, namely, there is only one pair of inertial observers, which are connected by these forms of transformations.

Although, the composition of two transformations in equation (3.1.16) is not valid to be the transformations that connect the third frame of reference, denoted by S'', to one of the previous frames of reference as we expect. Nevertheless, we are still able to use this step to determine the velocities addition formula[13].

By following the same steps in section (2.1.2), we then obtain

$$v'' = \frac{v + v'}{1 + Kvv'}. \tag{3.1.22}$$

From (3.1.21), there are three cases to be considered:

(1.) $K > 0$. Set $c = \sqrt{K}$, then equation (3.1.16) becomes

$$\begin{aligned} x' &= 1/\sqrt{(v/c)^2 - 1}\{x - vt\}, \\ t' &= 1/\sqrt{(v/c)^2 - 1}\{t - vx/c^2\}. \end{aligned} \tag{3.1.23}$$

(2.) For $k \le 0,$ there is no the real valued form of $a(v)$.

The equation (3.1.22) is the velocities addition law as discussed in section (2.2.2). Since we claim that both v and v' are superluminal or FTL, then v'' is subluminal or slower than light. This reflects the lacking of the closure property of this form of transformations as stated previously.

Suppose that $a(v) < 0$ for $v > 0$ then, we have $d(v) = a(v) < 0$ for $v > 0$. And note that $a(v)$ is an odd function on v. Let denote $a(v)$ as $\gamma(v)$, which can be expressed as

$$\gamma(v) = \begin{cases} -\dfrac{1}{\sqrt{(v/c)^2 - 1}} & ;\ c < v < \infty \\ \dfrac{1}{\sqrt{(v/c)^2 - 1}} & ; -\infty < v < -c. \end{cases} \tag{3.1.24}$$

We call $\gamma(v)$ in this form as the "Extended Lorentz Factor".
Suppose that $v > 0,$ then we have $a(v) < 0$, $d(v) < 0$, and $c(v) = -(1/v)(a^{-1}(v) + a(v)) > 0.$

$$\left.\begin{aligned} \frac{\partial x'}{\partial x} &= a(v) < 0, \\ \frac{\partial t'}{\partial t} &= d(v) < 0, \\ \frac{\partial x}{\partial x'} &= \frac{d(v)}{a(v)[d(v) + vc(v)]} > 0, \\ \frac{\partial t}{\partial t'} &= \frac{1}{d(v) + vc(v)} > 0. \end{aligned}\right\}$$

Remark: The set of all allowed relative velocities between two inertial frames is

$\Gamma^* = (-\infty, -c) \cup (c, \infty).$

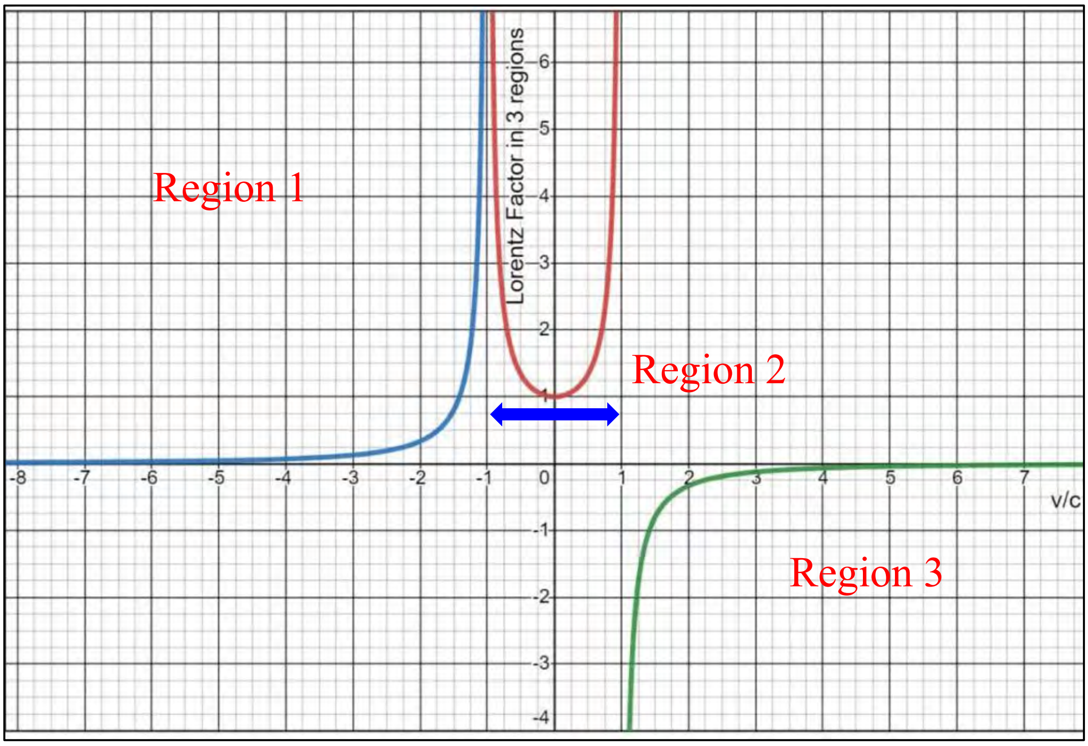

Figure 3.1: Lorentz factor(s) in three velocity regions.

3.2 Revision of Hill-Cox Transformations

In section (2.2.1), we discussed the derivations of ELT introduced by Hill-Cox. In this section we will provide a rigorous analysis of Hill-Cox transformations, which will be shown that they can be reduced into one equivalent form. Furthermore, we will point out that the so-called transformation is deeply associated with the anisotropy of space, and it is also consistent with the results we obtained in section (3.1).

Recall an autonomous system,

$$\left[1-\left(\frac{v}{c}\right)^2\right]\frac{dx'}{dv}=-t', \qquad \left[1-\left(\frac{v}{c}\right)^2\right]\frac{dt'}{dv}=-\frac{x'}{c^2},$$

with parameter ε such that

$$\frac{d}{d\varepsilon}=\left[1-\left(\frac{v}{c}\right)^2\right]\frac{d}{dv}.$$

By substituting $v = c\sin\phi$, we have

$$\frac{1+v/c}{1-v/c} = \left(\frac{1+v_0/c}{1-v_0/c}\right)e^{2\varepsilon/c}.$$

If we suppose that $v = v_0$ at $\varepsilon = 0$. In this section[13], we focus on the case that $v_0 \to \pm\infty$. That is,

$$v = c\coth\left(\frac{\varepsilon}{c}\right), \quad c \le |v| < \infty. \tag{3.2.1}$$

From equation (3.2.1) and the identity

$$\operatorname{csch}^2\left(\frac{\varepsilon}{c}\right) = \coth^2\left(\frac{\varepsilon}{c}\right) - 1,$$

one obtains

$$\sinh\left(\frac{\varepsilon}{c}\right) = \pm\frac{1}{\sqrt{\frac{v^2}{c^2}-1}}, \qquad \cosh\left(\frac{\varepsilon}{c}\right) = \frac{|v|/c}{\sqrt{\frac{v^2}{c^2}-1}}. \tag{3.2.2}$$

From **Figure 3.2**, we see that $\varepsilon > 0$, then $v/c > 1$. Note that in this case, $v \in (-\infty, -c) \cup (c, \infty)$.

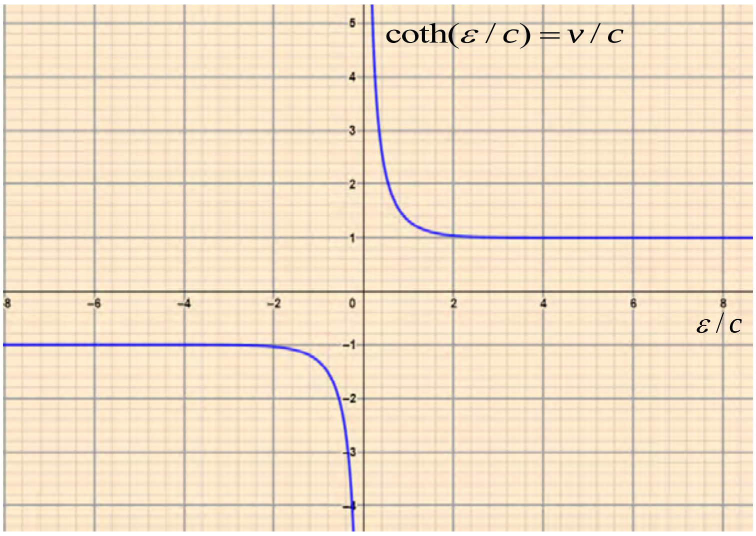

Figure 3.2: Graph of v/c versus ε/c

[13]The limit $v_0 \to \pm\infty$ indicates the superluminal relative velocity, i.e., $|v| > c$.

Thus,

$$\sinh\left(\frac{\varepsilon}{c}\right)=\begin{cases}\dfrac{1}{\sqrt{\dfrac{v^2}{c^2}-1}}; & c<v<\infty\\ -\dfrac{1}{\sqrt{\dfrac{v^2}{c^2}-1}}; & -\infty<v<-c.\end{cases} \tag{3.2.3}$$

We now define the special function as

$$sign(v)=\begin{cases}1; & c<v<\infty\\ -1; & -\infty<v<-c.\end{cases} \tag{3.2.4}$$

So we can write

$$\sinh\left(\frac{\varepsilon}{c}\right)\equiv\sinh(v)\equiv\frac{sign(v)}{\sqrt{\dfrac{v^2}{c^2}-1}}. \tag{3.2.5}$$

We further assume that the formula of velocities addition is still applicable to this case, from section (2.2.2), we know that $u\cdot u'=c^2; v\to\pm\infty$. By solving the second order differential equations as in section (2.2.2), we have

$$\begin{aligned} x'(\varepsilon)&=A\sinh\left(\frac{\varepsilon}{c}\right)+B\cosh\left(\frac{\varepsilon}{c}\right),\\ t'(\varepsilon)&=-\frac{[A\cosh(\varepsilon/c)+B\sinh(\varepsilon/c)]}{c}.\end{aligned} \tag{3.2.6}$$

We can consider as follows:

When $\varepsilon\to 0^{\pm}, v\to\infty$, $x'=B, t'=-A/c$. Recall the condition $u\cdot u'=c^2; v\to\pm\infty$, we can write $u=\dfrac{x}{t}, u'=\dfrac{x'}{t'}$. We also expect that $uu'=\left(\dfrac{-cB}{A}\right)\left(\dfrac{x}{t}\right)=c^2$, which implies $\dfrac{x}{ct}=-\dfrac{A}{B}$. We also assume that $\mathrm{mod}(A,B)=0,$ then we can write $A=\pm x,\ B=\mp ct.$

Thus, we have the transformations

$$x'=\frac{1}{\sqrt{\dfrac{v^2}{c^2}-1}}[\pm sign(v)\cdot x\mp|v|t].$$

By substituting $|v| = sign(v)\cdot v,$ then we can write

$$x' = \frac{sign(v)}{\sqrt{\frac{v^2}{c^2}-1}}[\pm x \mp vt],$$

$$x' = \frac{\pm sign(v)}{\sqrt{\frac{v^2}{c^2}-1}}[x - vt]. \tag{3.2.7}$$

Similarly, we can write

$$t' = \pm\frac{sign(v)}{\sqrt{\frac{v^2}{c^2}-1}}(t - vx/c^2). \tag{3.2.8}$$

Now, we may define

$$\gamma(v) = \pm\frac{sign(v)}{\sqrt{\frac{v^2}{c^2}-1}}. \tag{3.2.9}$$

We can equivalently determine the value of $\gamma(v)$, which depends on the sign of v, previously we chose that $v > 0,\ \gamma(v) < 0$. Therefore, we can write the extended Lorentz factor as

$$\gamma(v) = -\frac{sign(v)}{\sqrt{\frac{v^2}{c^2}-1}},$$

or

$$\gamma(v) = \begin{cases} -\dfrac{1}{\sqrt{(v/c)^2-1}} & ;\ c < v < \infty \\ \dfrac{1}{\sqrt{(v/c)^2-1}} & ;-\infty < v < -c. \end{cases} \tag{3.2.10}$$

The analyses above, both in sections (3.1) and (3.2), are not presented in the original work of Hill-Cox. However, these results reflect how the particular form of ELT is related to anisotropy of space, and its corresponding peculiar physical interpretation.

3.3 Geometry of Spacetime and Physical Interpretation Corresponding to ELT

In this section, we first consider the inverse Hill-Cox transformations, by considering the geometry of light cones under these transformations, to obtain the results in section (3.2). The procedures we use in the first part are modified by the steps in section (2.3). Moreover, we can provide a new physical interpretation of the propagation of light, by considering the geometry of spacetime associated with the ELT.

In section (2.3), we discussed the deformation of light cones by HC_1 and HC_2, by writing

$$HC_i = \rho^{-1} \circ \sigma_i \circ L \circ \rho, \text{ for } i = 1, 2.$$

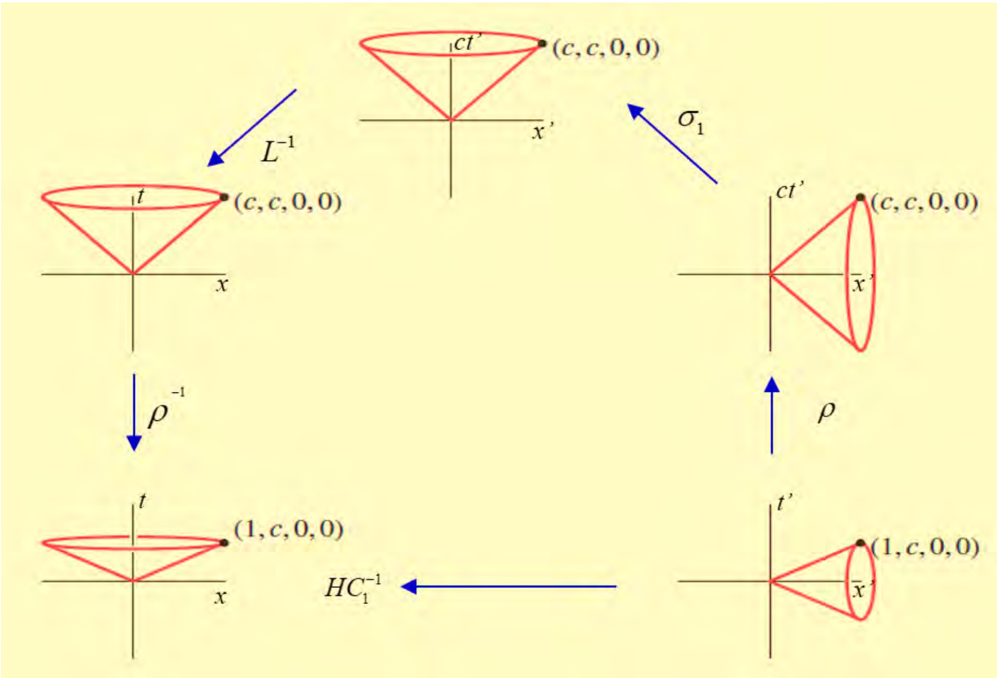

Figure 3.3: Decomposition of inverse Hill-Cox transformations
(Modification is based on Andreka *et al*, Proc. R. Soc. A, 469, 2154 (2013))

We have shown earlier that these two transformations, i.e., HC_1 and HC_2, are equivalent to be applied. Nevertheless, if we choose one of them, say HC_2, the other one can be obtained by reverse the direction of the relative velocity v between two inertial frames. This property is important because they are deeply related to the fact that the space of either inertial observers is not isotropic, Hill-Cox regarded these two transformations as they are independent, by merely reversing the orientations of spatial and time directions. The main point is the spatial direction cannot be switched, but rather the orientation of time can be reversed, which is distinct from the usual Lorentz transformations.

Consider the inverse transformation of HC_2 by writing

$$HC_1^{-1} = \rho^{-1} \circ L^{-1} \circ \sigma_1 \circ \rho, \tag{3.3.1}$$

where $\rho^{-1} : (ct, x, y, z) \mapsto (t, x, y, z),\quad L^{-1} : (t', x', y', z') \mapsto (t, x, y, z),$ such that

$$L^{-1} : t = \frac{t' + (c/v)x'}{\sqrt{1-(c/v)^2}}, \quad x = \frac{x' + (c/v)t'}{\sqrt{1-(c/v)^2}}, \quad y = y', \quad z = z'. \tag{3.3.2}$$

Now we can write

$$\begin{aligned}
\rho^{-1} \circ L^{-1} \circ \sigma_1 \circ \rho(t', x', y', z') &= (\rho^{-1} \circ L^{-1} \circ \sigma_1)(ct', x', y', z') \\
&= \left(\rho^{-1} \circ L^{-1}\right)(x', ct', y', z') \\
&= \rho^{-1}\left(\frac{x' + (c/v)ct'}{\sqrt{1-(c/v)^2}}, \frac{ct' + (c/v)x'}{\sqrt{1-(c/v)^2}}, y', z' \right) \\
&= \left(\frac{x'/c + (c/v)t'}{\sqrt{1-(c/v)^2}}, \frac{ct' + (c/v)x'}{\sqrt{1-(c/v)^2}}, y', z' \right) \\
&= \left(\frac{(c/v)(vx'/c^2 + t')}{(c/v)\sqrt{(v/c)^2 - 1}}, \frac{(c/v)(vt' + x')}{(c/v)\sqrt{(v/c)^2 - 1}}, y', z' \right) \\
&= \left(\frac{t' + vx'/c^2}{\sqrt{(v/c)^2 - 1}}, \frac{x' + vt'}{\sqrt{(v/c)^2 - 1}}, y', z' \right) \\
HC_1^{-1}(t', x', y', z') &= (t, x, y, z).
\end{aligned} \tag{3.3.3}$$

From section (2.3), we showed the steps of the forward HC_1 which acted on the tuple (t, x, y, z), the obtained result was

$$(t', x', y', z') = \left(\frac{-t + vx/c^2}{\sqrt{(v/c)^2 - 1}}, \frac{-x + vt}{\sqrt{(v/c)^2 - 1}}, y, z \right). \tag{3.3.4}$$

From the results of equations (3.3.3) and (3.3.4), one can see that the extended Lorentz factor has the opposite sign when the relative velocity is reversed. This is also consistent with the results, which we have already obtained in sections (3.1) and (3.2).

We also provide two crucial remarks at this step. First is that the commutator of the two operators L and σ_i is zero, or $[L,\sigma_i]=0$. The other point is about the operator σ_i, i.e., it has its inverse as itself because it acts on the first and second components of any tuples (of spacetime) by switching them (for σ_2 it also attaches the negative sign to the two switching components).

Consider the propagation of light in the frame of reference which has the light cone equation as

$$x^2=(ct)^2+y^2+z^2. \tag{3.3.5}$$

We denote ξ as (t,x,y,z) in this section.
We define $LC^* \equiv \{\xi : x^2=(ct)^2+y^2+z^2\}$ as the light cone corresponding to equation (3.3.5), $H_\tau^+ \equiv \{\xi : t=\tau \in \mathbb{R}^+\}$, and $H_\tau^- \equiv \{\xi : t=\tau \in \mathbb{R}^-\}$ as the time-hyperplane with positive and negative values of time coordinate respectively.

The propagation of light as observed in this frame can be described by $LC^* \cap H_\tau^+$ and $LC^* \cap H_\tau^-$, which can be interpreted as two photons move along the $x-$axis such that one of them moves forward in time, and the other moves backward in time. If we apply the switching principle, then we can say that the backward in time photon is just the forward in time anti-photon. Since an anti-photon is the same as a photon, then we can equivalently interpret that there are two photons that move along the $x-$direction at any instance of time. We may expect that they form the bound state or they perhaps interfere with each other. The obtained result is consistent with the derivation of [3] that the propagation of light that is restricted only along the $x-$direction, and we also proposed a new interpretation about the two-photon system.

In contrast, the propagation of light in the other frame, i.e., the frame which has the isotropic space, is the spherical surface in three-dimensional space. This can be explained by the following steps

Define $LC \equiv \{\xi : (ct)^2=x^2+y^2+z^2\}$ as the usual light cone, which is associated with the Lorentz transformations), and $H_\tau \equiv \{\xi : t=\tau \in \mathbb{R}^+\}$ is the time-hyperplane, since time in this case always flows in the same direction, say positive direction. Then we have $LC \bigcap H_\tau = S^2 \equiv \{\xi : x^2+y^2+z^2=(c\tau)^2\}$, as the spherical surface of light propagation in three-dimensional space.

There is a suggestion [1] that the propagation of light observed in the frame with the light cone LC^* should be in the hyperboloid of two sheets. We comment that the hyperboloid of two sheets surface can be obtained from the equation $x^2=(ct)^2+y^2+z^2$, then we may directly rearrange it as $z^2+y^2-x^2=-(ct)^2$, which becomes the hyperboloid of two sheets surface. In spite of the case of the light cone LC, we can obtain the equation of the propagation of light by those steps, but the argument above is not valid in general, because we cannot switch two coordinates, i.e., x and t directly from the equation of light cone to regard it as the equation of the propagation of light. The reason that those steps are equivalent to the method of applying $LC \cap H_\tau$ is due to the orientation of the light cone LC along the time axis, which is appropriate to intersect the time-hyperplane H_τ to form the spherical surface and this is equivalent to assign the value of t with τ and regard $c\tau$ as the radius of the sphere S^2 at any time τ.

3.4 The Relations between Subluminal and Superluminal Sets of Velocities

Define the two sets as $\Gamma \equiv (-c,c)$, $\Gamma^* \equiv (-\infty,-c) \cup (c,\infty)$, which corresponds to the subluminal set and superluminal set respectively.

We further claim that the three functions/mappings $\varphi : \Gamma \to \Gamma$, $\varphi^* : \Gamma^* \to \Gamma^*$, and $\tilde{\varphi} : \Gamma \setminus \{0\} \to \Gamma^*$ are continuous on their domains and they are bijective. Suppose that both φ and φ^* are associated with the functions of the relative velocity of two inertial frames as discussed in the derivations of LT and ELT. The new function $\tilde{\varphi}$ can be considered as the connection between the subluminal set and the superluminal set.

For $v \in \Gamma$; $\varphi(v) = -v$, and for $v \in \Gamma^*$; $\varphi^*(v) = -v$, these two relations satisfy the reciprocity principle. One can notice that both Γ and Γ^* are symmetric, i.e., if $v \in \Gamma$, then $-v$ is so, and the same for Γ^*. To determine the form of $\tilde{\varphi}$, the motivation from topology that any open interval of $\mathbb{R}$ is homeomorphic to other open interval of $\mathbb{R}$ is taken into account. Although the domain of $\tilde{\varphi}$ is open on $\mathbb{R}$, but it is not an interval of $\mathbb{R}$.

Suppose that $X = (-1,1)$, we can define a continuous bijective function $f : (-1,1) \to \mathbb{R}$ by

$$f(x) = \tan\left(\frac{\pi x}{2}\right). \tag{3.3.6}$$

Its continuous inverse is

$$f^{-1}(x) = \frac{2}{\pi}\arctan(x). \tag{3.3.7}$$

Therefore, $f : (-1,1) \to \mathbb{R}$ is a homeomorphism.

Now, we can modify the function $\tilde{\varphi} : \Gamma \setminus \{0\} \to \Gamma^*$, with some additional conditions. We construct the function $\tilde{\varphi}$ as

$$\tilde{\varphi}(v) \equiv \begin{cases} c + \tan\left\{\frac{\pi}{2}\left(\frac{v}{c}\right)\right\}; & 0 < v < c \\ -c + \tan\left\{\frac{\pi}{2}\left(\frac{v}{c}\right)\right\}; & -c < v < 0. \end{cases} \tag{3.3.8}$$

And the corresponding inverse is

$$\tilde{\varphi}^{-1}(v) \equiv \begin{cases} \frac{2c}{\pi}\arctan(v-c); & 0 < v < c \\ \frac{2c}{\pi}\arctan(v+c); & -c < v < 0. \end{cases} \tag{3.3.9}$$

The functions $\tilde{\varphi}$ and $\tilde{\varphi}^{-1}$ are continuous on their domains. Explicitly, the existence of the function $\tilde{\varphi}$ connects the two sets of velocity, i.e., the subluminal set and the superluminal set. We observe that if we write the function φ^* as $\varphi^* = \tilde{\varphi} \circ \varphi \circ \tilde{\varphi}^{-1}$, then we can suggest a particular configuration of inertial frames such that some of them can be transformed from the subluminal set to the superluminal by constructing a new pattern of transformation.

We define a two-sided arrow as the velocity connection between any two inertial frames such that the velocity mapping is the same as its inverse, e.g., $\varphi : \Gamma \to \Gamma$, and φ^{-1} is the same as φ, and is similar to $\varphi^* : \Gamma^* \to \Gamma^*$. But for the mapping $\tilde{\varphi} : \Gamma \setminus \{0\} \to \Gamma^*$, $\tilde{\varphi}^{-1}$ is distinct from the function $\tilde{\varphi}$, then we denote the velocity connection between two inertial frames associated with the function $\tilde{\varphi}$ with a one-sided arrow. We mention at this step that the dashed lines in **Figure 3.4** represent a velocity relation between any two inertial frames in the successive step.

In order to transform from the set Π to the set Π^* we need to assume either one of two conditions to satisfy the reciprocity principle and to apply the obtained ELT validly, i.e., we need to either assume that there exists one pair or many isolated pairs of inertial frames such that the two inertial frames in each pair is connected by the function φ^* and there is no connection among isolated pairs of inertial frames in the set Π^*, or there exists more than one pair of inertial frames in the set Π^*, but only the isotropic space-frame (S_i) can be connected to the anisotropic space-frame (S_i') via the function φ^*, and there is no connection among $\{S_i\}$, which applies the same to $\{S_i'\}$. We also remark that the two assumptions are necessary to maintain the fact that the ELT do not form a group, therefore the set of inertial frames which is associated with the function $\varphi^* : \Gamma^* \to \Gamma^*$ is likely to correspond to one of the two assumptions.

We insist on the more general case as depicted in **Figure 3.4** which shows a pathway of a transformation from a set Π to a set Π^* via intermediate processes involving the particular sets $\tilde{\Pi}_i$. In the set Π, the velocity relation for any pair of inertial frames is represented by the function $\varphi : \Gamma \to \Gamma$, namely, it belongs to the Minkowski spacetime. The intermediate step occurs when the set $\tilde{\Pi}_1$ is separated from the set Π. Then S_1 and $\tilde{S}_1$ are connected by the function $\tilde{\varphi}_1$, likewise for $(S_2, \tilde{S}_2)$, which is connected by $\tilde{\varphi}_2$. The inverse relations are denoted by the opposite one-sided arrow. It is reasonable to suppose that both frames S_1 and S_2 are associated with isotropic space, whereas the two frames $\tilde{S}_1$ and $\tilde{S}_2$ perceive anisotropic space. Then the set $\tilde{\Pi}_2$ emerges as the set that contains S_1, S_2 and S_5, which should be associated to isotropic space, since $\tilde{\Pi}_2$ can be viewed as the usual set Π. In the set $\tilde{\Pi}_2$, all frames are connected by the function φ, and there is no connection between any two frames in the set $\tilde{\Pi}_1$ at this step. Notice that now $\tilde{S}_1$ and $\tilde{S}_2$ are in an isolated set $\tilde{\Pi}_3$ such that they are not connected by the velocity function, i.e., it physically implies that they cannot observe each other. In the last step, S_1 and S_2 are still connected by the function φ and belongs to the set Π. The separated set Π^* contains three inertial frames $S_5, S_1^{'}$ and $S_2^{'}$, where $S_1^{'}$ and $S_2^{'}$ are the frames $\tilde{S}_1$ and $\tilde{S}_2$ respectively as re-labelled by prime notation in the set Π^*. In the set Π^* S_5 is conjugated with $S_1^{'}$ and $S_2^{'}$ by the function φ^*, and there is no connection between $S_1^{'}$ and $S_2^{'}$.

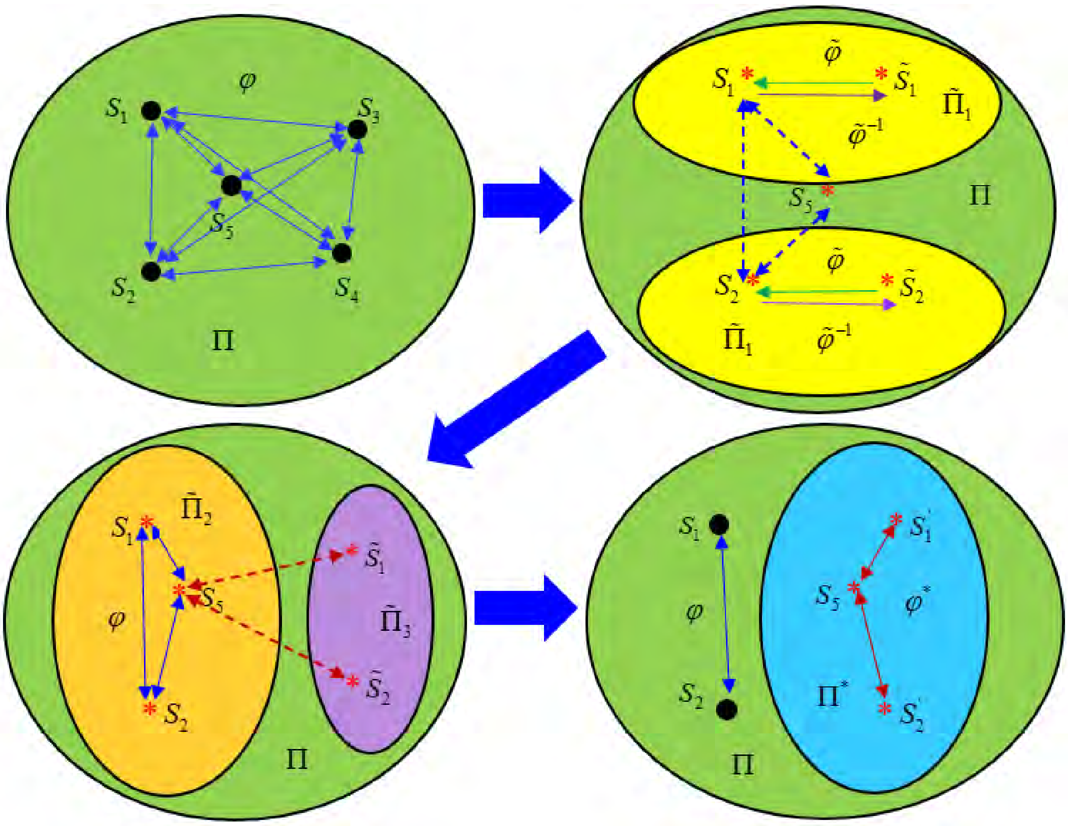

Figure 3.4: Transformations of the inertial frames' configuration from the set Π to the set Π^*

The physical interpretation of the result is rather peculiar, i.e., normally, one (at rest in the second frame) can use the velocity addition formula to deduce the relative velocity of the third frame compared to the second frame if the relative velocity between the first frame with respect to the second frame and the relative velocity between the first frame with respect to the third frame are known. However, in this case, the ELT which connect inertial frames in the set Π^* by a particular manner prohibits deducing the other relative directly by using the velocity addition formula, namely, only the relative velocity between any two inertial frames can be obtained directly by observation (at least in principle), this is the consequence that we need to maintain the validity of the ELT under the reciprocity principle which the mapping $\Gamma^* \to \Gamma^*$ must be restored. The main aspect of the relation in the set Π^* is that a predicted value by applying the velocity addition law is not consistent to a real observational value, i.e.,no observation that can be performed among the $\{S_i^{'}\}$ to determine the reference frame relative velocities.

3.5 (3, *n*)-Ultra-Hyperbolic Structure of Spacetime

In this section, we provide the special structure of spacetime to verify whether the validity of the principle of invariance of the speed of light is valid in this particular spacetime. By constructing the $(3,n)$-ultra-hyperbolic structure, which is composed of three spatial coordinates and n-coordinates of time. This is motivated by considering the propagation of light when the two inertial frames are conjugated with the ELT, as stated earlier, the speed of light is not (universally) invariant in this scenario. This problem can be mathematically solved by inserting two-time coordinates to form the six-dimensional spacetime as suggested by [1], the reason that this unique spacetime structure preserves the invariance of the speed of light is that each observer can access only four physical spacetime together with a unique way to access the space-time coordinates, i.e., by appropriate matching of spatial and time coordinates, of the same event as viewed by two inertial observers. One can see that inserting two extra dimensions of time results in the same number of spatial coordinates and time coordinates, this is the main point for applying the so-called unique way to access the space-time coordinates. We also comment that for n-coordinates of time with $n>3$, there must exist the imaginary time coordinates if we apply the previous procedure as in the case of six-dimensional spacetime.

We now put forward the $(3,n\geq 2)$-ultra-hyperbolic structure such that each observer can access only four-dimensional spacetime such that if one of n-time coordinates is accessed by an observer, the remaining $(n-1)$-time coordinates will no longer be approached, then the dynamics of any event (except a light-like event) as measured by that observer will correspond to only the accessible time coordinate. We also provide an additional assumption that only when light (photon) or a light-like event is observed, all n-degree of freedom of time coordinates are accessible to an observer who observes it.

Define the (differential) spacetime metric as

$$
\begin{aligned}
ds^2 &:= c^2(dt_1^2+dt_2^2+\ldots+dt_n^2)-(dx_1^2+dx_2^2+dx_3^2), \\
&\equiv c^2(dt_x^2+dt_y^2+\ldots+dt_n^2)-(dx^2+dy^2+dz^2).
\end{aligned}
\tag{3.5.1}
$$

For the case of the Lorentz Transformations, we can suppose that the translation of S' frame with respect to S frame is along the x and x' axes (for simplicity, they have the same orientation) such that S frame can access t_x and S' frame can access $t_x^{'}$. We can write the relation

$$ds^2 = ds'^2,$$

$$c^2(dt_x^2 + dt_y^2 + ... + dt_n^2) - (dx^2 + dy^2 + dz^2) = (dx'^2 + dy'^2 + dz'^2) - c^2(dt_x^{'2} + dt_y^{'2} + ... + dt_n^{'2}). \quad (3.5.2)$$

Where the Lorentz transformations connect (x, y, z, t_x) and $(x', y', z', t_x^{'})$, if any observer in either frame observes any event which is not light-like, the remaining $(n-1)$-time coordinates are not accessible to that observer. For the case of propagation of light, either observer can access n- degree of freedom of time coordinates, then both observers in S and S' frames should observe the same propagation of light, i.e., the invariance of the speed of light is valid. So that we may say, in this case, $(t_y,...,t_n) = (t_y^{'},...,t_z^{'})$. At this point, one may claim that we normally live in the universe which prescribes each observer to has only one time coordinate, i.e., the universe according to special relativity. Thus, in the case of the Lorentz transformations, one may set $(t_y,...,t_n) = (t_y^{'},...,t_z^{'}) = (0,...,0)$ to preserve the usual four-dimensional spacetime with Lorentz signature.

Now we consider the case that two inertial frames are connected by the ELT, and also suppose that the translation of the S' frame with respect to the S frame is along the x and x' axes. Since, the actual current models of ELT have never been tested in practice, and we aim to verify the possible mathematical structure of ELT which is proper to apply the invariance of the speed of light. This model is directly modified from the relation $ds^2 = ds'^2$ according to the ELT which we previously obtained in four-dimensional spacetime. We have the relation as in equation (3.5.2)

$$c^2(dt_x^2 + dt_y^2 + ... + dt_n^2) - (dx^2 + dy^2 + dz^2) = (dx'^2 + dy'^2 + dz'^2) - c^2(dt_x^{'2} + dt_y^{'2} + ... + dt_n^{'2}).$$

In this case, (x, y, z, t_x) and $(x', y', z', t_x^{'})$ are connected by the ELT. We further claim that to satisfy the principle of invariance of the speed of light we may assume that

$$(dy^2 + dz^2) + (dy'^2 + dz'^2) = c^2(dt_y^2 + dt_z^2 + ... + dt_n^2) + c^2(dt_y^{'2} + dt_z^{'2} + ... + dt_n^{'2}). \quad (3.5.3)$$

We then manipulate it as

$$\begin{aligned} dr^2 + dr'^2 &= c^2[(dt_y^2 + dt_z^2 + ... + dt_n^2) + (dt_y^{'2} + dt_z^{'2} + ... + dt_n^{'2})], \\ &\equiv c^2(d\tau^2 + d\tau'^2), \quad (3.5.4) \\ &\equiv d\rho^2 \geq 0, \end{aligned}$$

for any finite $n \in \mathbb{Z}^{+}, n \geq 2$.

Equation (3.5.4) is merely the circular equation of coordinates (r, r') with the radius of $\rho = c(\tau + \tau')$. Since we do not derive this particular spacetime structure from the fundamental axioms, but we rather extend it from the obtained four-dimensional spacetime with ELT signature to recover the problem of propagation of light as viewed by two different observers. Therefore, we comment that, for convenience, the flows of $\{t_y, t_z, ..., t_n\}$ are (strictly) monotonic increasing, this also applies to $\{t_y^{'}, t_z^{'}, ..., t_n^{'}\}$. Hence, the radius $\rho \geq 0$. Note that only t_x and $t_x^{'}$ which are conjugated with ELT, and their orientations are not unity. The relation in equation (3.5.4) is based on the assumption that when a light-like event is observed from either observer, an observer must access all n-degree of freedom of time coordinates.

According to the analysis by the intersection of the time-hyperplanes to both light cones associated to S and S' frames in n-dimensional spacetime, we found the consistency of the light phenomena as observed by two inertial observers, i.e., the observer in S frame (isotropic space) always views the propagation of light in the spherical shape in spatial dimensions with the radius of $c^2 d\tau^2$, likewise, the observer in S'frame should also observe the spherical propagation of light with the radius of $c^2 d\tau'^2$. This result is obviously due to the invariant relation of $ds^2 = ds'^2$, then one may trivially see from a rewritten form

$$c^2(dt_x^2 + dt_y^2 + ... + dt_n^2) - (dx^2 + dy^2 + dz^2) = (-)[c^2(dt_x^{'2} + dt_y^{'2} + ... + dt_n^{'2}) - (dx'^2 + dy'^2 + dz'^2)].$$

The equation above reflects the non-preserved partial ordering of any two events on the set Π^* in which the function $\varphi^* : \Gamma^* \to \Gamma^*$ is the velocity mapping of a pair of two inertial frames, i.e., suppose that the observer who at rest in the S frame measures an event in spacetime as a time-like event, then the other observer in the S'frame classify the identical event to be a space-like event, and vice versa. The two observers would agree on the type of event if and only if that event is a light-like event.

Chapter 4

Conclusion

We have derived the ELT by considering the symmetries of space and time, and we also confine the derivation under the reciprocity principle. The obtained transformations are inherently associated with the invalidity of the principle of the speed of light, which one may extend this scenario to the violation of the principle of relativity. However, these types of transformations were similarly proposed by Hill and Cox [2]. In the present project, we derive ELT from the fundamental footings of space and time, so that we can verify some interesting consequences from this point. The modification of Hill-Cox transformations has been provided, which are consistent with the derived ELT. We then considered the decomposition of the obtained ELT as guided by the analysis introduced by [3], in this section, we provided the consideration of the propagation of light by performing the intersection of the time-hyperplanes corresponding to both inertial frames' time coordinates to their light cones. The result is partially consistent with the analysis of [3], but we suggested a new way to interpret the propagation of light as viewed by two different superluminal observers. Furthermore, there are three distinct continuous (on their velocity domains) bijective functions, which can be regarded as the mappings of the direct velocities to the reciprocal velocities. The functions $\varphi:\Gamma\to\Gamma$ and $\varphi^*:\Gamma^*\to\Gamma^*$ are symmetric and they correspond to the two sets Π and Π^* respectively, whereas the function $\tilde{\varphi}:\Gamma\setminus\{0\}\to\Gamma^*$ is not symmetric, and its inverse is different from itself. The function $\tilde{\varphi}$ connects the pathway to transform the set Π to the set Π^* by forming two intermediate steps. We claim that one can probably transform the set Π^* to the set Π by applying the inverse (backward) procedure. Certainly, this can be achieved due to the fact that all velocity mappings/functions are bijective, and by writing in the composition form of them, one may obtain both forward and backward transformations. Moreover, we suggest the unique configuration of the set of inertial frames, which are connected by ELT. By reviewing the dynamics in which are associated with SR and ESR, and by reviewing the previously suggested six-dimensional spacetime [1] to recover the invariance of the speed of light, we then proposed the (3, *n*)-ultra-hyperbolic structure of spacetime under the assumption that only when an observation of light is made in either frame (of reference), an observer in that frame can access all *n*-degrees of freedom of his time coordinates to maintain the spherical propagation of light in spatial dimensions (3D). In addition, the dynamics corresponding to the (3, *n*)-ultra-hyperbolic scheme may possibly be described by the dynamics which are associated with ESR, namely, together with ELT signature, such that the extra assumption of time coordinates accessibility is the constraint.

Appendices

Appendix A

Standard Proof Related to the Homogeneity of Space-time

Let g be a mapping of $\mathbb{R}^n \to \mathbb{R}^n$ such that

$$g(\xi+\delta)=g(\xi)+g(\delta). \tag{A1}$$

If n is a positive integer, we then apply the mathematical induction, from (A1), we obtain

$$g(n\xi)=ng(\xi). \tag{A2}$$

Recall that $g(\xi)=f(\xi)-f(0),$ therefore, $g(0)=0.$

Let $\delta=-2\xi,$ from equation (A1), we obtain

$$g(-\xi)=g(\xi)+g(-2\xi)=g(\xi)+2g(-\xi).$$

Thus,

$$g(-\xi)=-g(\xi).$$

For any rational $r=m/n,$ and set $m\xi=n\lambda.$ Then one would obtain

$$mg(\xi)=g(m\xi)=g(n\lambda)=ng(\lambda),$$

and thus,

$$g(r\xi)=rg(\xi). \tag{A3}$$

Suppose that the function g is continuous at the origin, that is the limit exists at $\xi=0.$

$$\lim_{\xi\to 0^+} g(\xi)=\lim_{\xi\to 0^-} g(\xi)\equiv \lim_{\xi\to 0} g(\xi)=g(0).$$

Now consider

$$\begin{aligned}\lim_{\xi\to 0} g(\xi+\delta) &= \lim_{\xi\to 0} g(\xi)+g(\delta),\\ \lim_{\xi\to \delta} g(\xi) &= g(\delta).\end{aligned}$$

Therefore, the function g is continuous at every $\xi\in\mathbb{R}^n.$

Then, let k be any real number and $\{k_n\}$ be a sequence of rational numbers which converges to k. So,

$$\lim_{n\to\infty} k_n\xi \to k\xi.$$

Appendix A is elaborated from Appendix A presented in [7].

And by the continuity,

$$\lim_{n\to\infty} g(k_n\xi) = \lim_{n\to\infty}\left\{\lim_{\xi\to k_n\xi} g(\xi)\right\}$$
$$= \lim_{\xi\to k\xi} g(\xi) \to g(k\xi).$$

But

$$g(k_n\xi) = \lim_{n\to\infty} k_n g(\xi) \to kg(\xi).$$

Therefore,

$$g(k\xi) = kg(\xi). \qquad \text{(A4)}$$

Appendix B

Discussion of Sub-Superluminal Transformations and Group Property

Define $\Pi \equiv \{\Lambda_1, \Lambda_2, ..., \Lambda_k, ..\}$, such that identity operator $I \in \Pi$. From the reciprocity principle, we have $\Lambda_i^{-1}(v) = \Lambda_i(-v) \in \Pi$. We construct a pair $\Omega \equiv (\Pi, \odot)$ as a group under a binary operation $\odot$ which operates on any two elements of the set . One can show that Ω is indeed a group under $\odot$ by satisfying four axioms of being a group. Similarly, suppose we define $\Pi_i^* \equiv \{\Lambda_i^*, I, (\Lambda^*)_i^{-1}\}$. It is certainly a group under a binary operation $\odot$, denoted by $\Omega^* \equiv (\Pi^*, \odot)$. Now we extend to the definition of the a $\Pi^* \equiv \{\Pi_i^*\}$ as the collection of each group Π_i^* without containing an operator which operates on any two elements of the set. Technically, $\Omega \equiv (\Pi, \odot)$ is called the **Lorentz Group**, so we may call $\Omega^* \equiv (\Pi^*, \odot)$ as the **Extended Lorentz Group**. However, the physical interpretation of Ω^* is peculiar, because it implies the existence of an isolated pair of inertial frames (of reference).

In section 3.4, we provided the transformations of the inertial frames' configuration from the set Π to the set Π^* as shown in **Figure 3.4**. This is the modification of the **Extended Lorentz Group** Ω^* to the set $\Pi^* \equiv \{\Pi_i^*\}$. The physical interpretation of Π^* is rather fantastical, but we mainly confine ourselves to the mathematical structure which is extended from the standard special relativity together with the fundamental properties of space-time transformations and some additive mathematical arguments.

Note that Λ_i and Λ_i^* represent the transformation matrices correspond to the functions φ and φ^* respectively.

Appendix C

Connected Property of the set Γ^*

We, in this section, provide a proof that the set $\Gamma^* = (-\infty, -c) \cup (c, \infty)$ is the connected space by applying some propositions of general topology to obtain the so-called argument. Recall the proposition 2: Let $A \subseteq \mathbb{R}$, and A is connected if A is an interval, our task is to verify that Γ^* is whether an interval on $\mathbb{R}$.

Lemma 1: A set $X \subseteq \mathbb{R}$ is an interval exactly when it satisfies the following proposition:

Proposition 3: If $x < z < y$, and $x \in X$, $y \in X$, then $z \in X$.

One can obviously notice that Γ^* does not satisfy the proposition 3, thus Γ^* is not an interval on $\mathbb{R}$. However, we cannot directly conclude that Γ^* is not connected by only arguing that it does not form an interval on $\mathbb{R}$. Thus, we need more lemma, which states

Lemma 2: A subset of $\mathbb{R}$ is connected if and only if it's either of the following:

1.) Bounded: $(a,b),(a,b],[a,b),[a,b];$ for any $a,b \in \mathbb{R}.$

2.) Unbounded: $(-\infty,a),(-\infty,a],(a,\infty),[a,\infty);$ for any $a,b \in \mathbb{R}.$

According to the lemma 2, we can obviously classify that the set $\Gamma^* = (-\infty, -c) \cup (c, \infty)$ is not connected on $\mathbb{R}$.

Appendix D

Six-Dimensional Spacetime Proposed by Ricardo S. Vieira

Ricardo S. Vieira provided the section 11 of his paper [1] entitled “A possible theory in six dimensions” describing the possible mathematical structure of six-dimensional spacetime. We, in this appendix, review the proposed spacetime structure and provide a concise description in the case of imaginary time coordinates when one introduces more than three degrees of freedom of time coordinates.

Define the six-dimensional spacetime metric as

$$ds \equiv \sqrt{\left|c^2 dt_x^2 + c^2 dt_y^2 + c^2 dt_z^2 - dx^2 - dy^2 - dz^2\right|}, \tag{D1}$$

and the physical four-dimensional spacetime metric as

$$d\sigma \equiv \sqrt{\left|c^2 dt_x^2 - dx^2 - dy^2 - dz^2\right|}. \tag{D2}$$

Where we suppose t_x as the physical time measured by the observer in the S frame.

Let a particular event has coordinates in S as $(ct_x, ct_y, ct_z, x, y, z)$, but suppose that the observer can only access the coordinates (ct_x, x, y, z). Similarly, for the reference frame S', let the coordinates of the same events are $(ct_x^{'}, ct_y^{'}, ct_z^{'}, x, y, z)$ and only the coordinate $(ct_x^{'}, x, y, z)$ are accessible to S'.

In the case of the relative velocity (speed) between two reference frames $v < c$. It is possible to conclude that the coordinates (ct_x, x, y, z) are connected to $(ct_x^{'}, x, y, z)$ by the standard Lorentz transformations, and we can evidently write

$$ct_y^{'} = ct_y,\ ct_z^{'} = ct_z,\ y' = y,\ z' = z. \tag{D3}$$

Now consider the case of $v > c$, we now focus on the propagation of light in both perspectives of the two observers. In the case of four-dimensional ELT, the invariance of the speed of light is not valid. In this scenario, we rather impose the stronger hypothesis on the nature of the propagation of light in terms of the spacetime metric. Instead of confine the theory on the relation $d\sigma' = d\sigma = 0,$ we then apply the relation $ds' = ds = 0$.

One can verify that if we write

$$t_x^{'} = \pm \frac{t_x - vx/c^2}{\sqrt{(v/c)^2 - 1}},\ x' = \pm \frac{x - vt_x}{\sqrt{(v/c)^2 - 1}}, \tag{D4}$$

and the possible relations as

$$ct_y^{'} = \pm y,\ ct_z^{'} = \pm z,\ y' = \pm ct_y,\ z' = \pm ct_z. \tag{D5}$$

One may alternatively write as

$$ct_y^{'} = \pm z,\ ct_z^{'} = \pm y,\ y' = \pm ct_z,\ z' = \pm ct_y. \tag{D6}$$

We further claim that if we introduce more than three-time coordinates, some of the time coordinates must be imaginary to maintain the relation of propagation of light, i.e., $ds' = ds = 0$. However, this situation should not be taken into account, since we, in this context, aim to determine the possible mathematical real (coordinates) structure of spacetime with ELT signature, and is compatible with the principle of the invariance of the speed of light.

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