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COSMIC MICROWAVE BACKGROUND TESTS OF INFLATION MODELS

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วัตถุประสงค์ของวิทยานิพนธ์นี้คือการปริทัศน์การทดสอบแบบจำลองอินเฟลชันด้วยข้อมูลจากการสังเกตการณ์คลื่นคอสมิกไมโครเวฟแบ็กกราวน์ การเกิดของคลื่นคอสมิกไมโครเวฟแบ็กกราวน์และแอนไอโซทรอปีที่เกี่ยวข้องกับอินเฟลชันได้ถูกพิจารณา ในแบบจำลองอินเฟลชันที่เกิดจากตัวแปรสนามหนึ่งตัว ค่าสเปกตรัมกำลัง ค่าดัชนีสเปกตรัม และการไหลของค่าดัชนีสเปกตรัมของความแปรปรวนระยะแรกเริ่มได้ถูกคำนวณทั้งในแบบจำลองอินเฟลชันแบบสลับที่และไม่สลับที่ ในแบบจำลองอินเฟลชันที่เกิดจากตัวแปรสนามหลายตัวโดยใช้ศักย์แบบสมมาตรยวดยิ่ง ค่าสเปกตรัมกำลังของความแปรปรวนแบบเคอรเวเจอร์ ความแปรปรวนแบบไอโซเคอรเวเจอร์ และสหสัมพันธ์ได้ถูกคำนวณ

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The aim of this thesis is to review the test of slow-roll inflation models by using cosmic microwave background (CMB) observations. CMB formation and its anisotropy associated with inflation are considered. In the single-field inflation model, the power spectrum, spectral index and running of the spectral index of the primordial perturbations are computed in both commutative and noncommutative inflation models. In the multi-field inflation model with the supersymmetric potential, the power spectra of the curvature perturbation, the isocurvature perturbation and the correlation are computed.

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CHAPTER I

INTRODUCTION

Inflation theory was first proposed by Guth in 1981 [1]. Inflation is a period of accelerated expansion in the early universe. It occurs when the energy density of the universe is dominated by the potential energy of some scalar field called *inflaton*. During this period, the universe has accelerated expansion, and the quantum fluctuations are enlarged to be the primordial density fluctuations. They grow into the large-scale structure in the universe today. In order that inflation can occur, the slow roll approximation (the inflaton rolls very slowly along the almost flat potential) is needed to have a large amount of inflation to solve problems arising from the standard Big Bang model as well as the large scale of perturbations [1]-[3], [12]. Inflation ends when the inflaton comes to a minimum of the potential and oscillates about it. During the damped oscillations, the energy lost of the inflaton reheats the universe and all the ordinary matters are created.

The cosmic microwave background radiation (CMB) was discovered in 1964 by Penzias and Wilson [4]. It is the roughly isotropic radiation. Its spectrum is almost the blackbody spectrum of temperature about 2.725 K. However, it has small variations of temperature. The degree of anisotropies in CMB is about one part in 100,000 [2]. The origin of anisotropies in CMB is thought to be associated with the density fluctuations at the decoupling time, the time when the universe cooled down enough for protons and electrons to form neutral Hydrogen atoms (380,000 years after big bang). These fluctuations are the primordial density fluctuations generated from quantum fluctuations during inflation period. We study CMB anisotropies by using the power spectrum which indicates the amplitude of variations of the temperature as a function of the angular scale. For inflation, the shapes of the theoretical power spectrum depends on the parameters assumed by the model. In each model, the power spectrum and its derivatives (the spectral index and the running of the spectral index) are computed as the appropriate inflationary observables and compared with the present CMB observations.

The aim of this thesis is to test and discriminate different slow-roll inflation models by using CMB observations. The thesis is organized as follows. In Chapter II, CMB formation and its anisotropy associated with inflation are considered. Structure formation, all fluctuation scales and CMB power spectrum are discussed respectively. We end this chapter by showing the recent CMB data.

Chapter III concerns dynamics of inflation as well as the slow roll conditions: the conditions required for inflation. The following section presents cosmological perturbations in the form of the perturbed Einstein equation. In the last section, we compute the power spectrum of the primordial perturbations and its derivatives in terms of the slow roll parameters.

In Chapter IV, the inflation models driven by a single scalar field are studied. The commutative inflation and its problems will be in our interest. Noncommutative inflation model [13]-[21] is described as a candidate of solving the problems. The perturbation equation of motion and the power spectrum in Chapter III are modified due to the existence of the noncommutative parameter. All parameters of the noncommutative power-law inflation are constrained by the observational data and the results will be seen to be in good agreement with the observations.

Multi-field inflation models [29, 31] are studied in Chapter V. The classification of the primordial perturbations are included here. We continue to compute the evolution of multiple scalar fields. The simplest model called double inflation [25, 28, 30] is then discussed, as well as the correlation between perturbations [24]-[28]. In the last section, the realistic double inflation model in supersymmetric theory [28, 30] and some constrains due to WMAP data [27, 30] are described.

Finally, we draw our conclusion in Chapter VI.

CHAPTER II

COSMIC MICROWAVE BACKGROUND ANISOTROPIES

There are many models explaining the origin and the evolution of the universe. Inflation theory is one of the possible candidates. Which one is the best model of inflation? The inflationary parameters, as well as the cosmological parameters, are constrained by the recent data from Wilkinson Microwave Anisotropy Probe (WMAP). One of the WMAP results is the observed cosmic microwave background (CMB) anisotropies. Before constructing any inflation models, it is important to study the CMB observations.

2.1 Formation of the CMB and Its Properties

Cosmic microwave background radiation is the primordial light we can observe from every direction today. Photons are formed after the inflation period in the radiation era (when the age of the universe is less than one second). When the universe has cooled down enough for protons and electrons to form neutral hydrogen atoms in the matter era, photons have no charged particles to scatter, then they move freely to us as CMB.

The universe begins with the gigantic explosion called *Big Bang*. At that time the universe is very hot and dense, and when it expands it becomes cooler and less dense. The universe is a hot plasma containing particles such as electrons, protons, neutrons and photons. Photons interact with electrons by Thomson scattering[1] . When the temperature of the universe is about 3000K, atoms are formed, mostly hydrogen. There are no free electrons for photons to scatter, then they

[1] Thomson scattering being the Compton scattering in low temperature is the scattering of the electromagnetic wave by a point charge, and the incident wave has the same wavelength

travel freely in the direction of last scattering. Their wavelengths are stretched to be in microwave range owing to the expansion of the universe. The universe is not a plasma any longer, photons do not couple with matter anymore, so it becomes transparent. This time is called *decoupling time* occurred when the age of the universe is about 380,000 years old (in the matter-dominated period).

CMB is the most perfect black-body spectrum we know; however, its temperature is slightly different on different patches of the sky to 1 part in 100,000. This is called *anisotropies* [8, 9] which are caused by the variation of frequency as photons move into and out of more dense and less dense regions, and the Doppler effect (the photon is red shifted or blue shifted when the electron it scattered off is moving away or toward us).

The origins of the fluctuations in the density at the decoupling time are quantum fluctuations in the early universe. Next section, CMB anisotropies and their connection to inflation will be considered.

2.2 Anisotropy Mechanisms and Inflation

From the previous section, one of the important sources of CMB anisotropies is the gravitational potential fluctuations at the time CMB is formed. The photon gains energy (blueshifting) when it falls into the gravitational potential wells, and loses energy (redshifting) when it climbs out of the gravitational potential wells.

The gravitational potential fluctuations at the decoupling time come from the very tiny fluctuations in the early universe. According to the Heisenberg Uncertainty Principle, there are quantum fluctuations[2] created all the time. According to the observations [5], the age of the universe today is about 13.7 billion years. The age is too short for quantum fluctuations ($l \sim 10^{-35}$ m.) to grow to be galaxies or clusters of galaxies ($l > 10^{24}$ m.) Inflation, which begins when the age of the universe is less than one second (about Planck time), can be one of

---

with the scatter wave. The Thomson scattering cross section of photons by electron is

$$\sigma_T = \frac{8\pi\alpha^2}{3m^2}$$

where m is the electron mass and $\alpha = \frac{e^2}{2\pi c\hbar}$ is the fine structure constant.

[2]Quantum fluctuations can be classified into two kinds:

- Fluctuations in the inflaton field: $\phi(t, \vec{x}) = \phi(t) + \delta\phi(t, \vec{x})$.
- Geometrical perturbations of the spacetime metric: $\mathrm{g}_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}$.

the answers. Inflation enlarges fluctuations on quantum scales into cosmic scales. Because there are all scales of fluctuations, after inflation, classical fluctuations can be separated into three scales: large scale fluctuations which are the source of galaxies and clusters of galaxies, intermediate scale fluctuations, and small scale fluctuations.

After inflation, the enlarged fluctuations are considered in the form of gravitational potential wells (more dense regions) and gravitational potential hills (less dense regions) in the plasma.

Gravity compresses all matters in the plasma into potential wells, but the photon pressure coming from the scattering of electrons resists the gravity. These events cause acoustic oscillations or sound waves. When the sound wave causes compression, the plasma gets more dense and hotter, but when it causes rarefaction, the plasma gets less dense and colder. From inflation, there are many scales of fluctuations, so there are different modes of oscillation in different length scales. Each mode oscillates independently[3]. The frequency of oscillation is kc_s where c_s is the sound speed. The oscillations tell us that there is the changing in time between the less dense state and the more dense state in each region.

Acoustic oscillations stop at the decoupling time when electrons are combined into atoms, so photons bound with nothing. They travel freely to us as CMB. Modes that reach the extrema of their oscillations will increase the difference in the energy density, and also the temperature. The patterns at the end of the oscillations are sent to us as the CMB anisotropies.

2.3 Structure Formation and Fluctuation Scales

After the decoupling time (all modes stop oscillating), gravitational fluctuations are amplified by gravity and grow into structure observed today.

Different scales of fluctuations cause different scales of structure.

2.3.1 Large Angular Scales ($\theta \geq 10^\circ$)

Large scale fluctuations (enlarged by inflation) are the modes in which length scales are much larger than the wavelength of the sound wave. These modes will not oscillate until the decoupling time. Then the fluctuations amplitudes are unchanged and become the large scale anisotropies in CMB, and will grow

[3] if the fluctuations are Gaussian.

to be galaxies and clusters of galaxies. Because the large scale fluctuations come from the quantum fluctuations in the early universe, they tell us about the initial conditions of the universe.

2.3.2 Intermediate Angular Scales ($1^\circ \leq \theta < 10^\circ$)

Intermediate scale fluctuations whose wavelengths are of the order of the sound horizon have acoustic oscillations. The oscillations stop at the decoupling time and leave the fluctuations in the new patterns which cause intermediate scale anisotropies in CMB.

This scale of the anisotropies corresponds to the scale in the CMB power spectrum in the range $100 \leq l < 1000$.

2.3.3 Small Angular Scales ($\theta \ll 1^\circ$)

We know that the decoupling time is not instantaneous, photons are able to diffuse out of the more/less dense regions and balance their temperature. This event reduces photons temperature variations, and decays the fluctuations amplitudes during the oscillations. So, the anisotropies in the small scale have the *Silk damping* as we see in the power spectrum for $l \geq 1000$.

We see that only the large scale perturbations bring the information of inflation directly to us because they neither oscillate, nor damp.

2.4 CMB Power Spectrum

CMB power spectrum shows the size of variations of the temperature as a function of the angular scale. Since temperature fluctuations distribute over the surface of the sphere, it is convenient to write them in terms of the spherical harmonics. One introduces the temperature fluctuation $\Theta(\hat{n}) \equiv \frac{\triangle T}{T}$ [6]:

$$\Theta(\hat{n}) = \sum_{lm} a_{lm} Y_{lm}(\hat{n}). \tag{2.1}$$

In this work, we consider only the Gaussian random fluctuations whose statistic properties give

$$< a^*_{lm} a_{l'm'} >= \delta_{ll'} \delta_{mm'} C_l. \tag{2.2}$$

In addition, the temperature perturbation field can be described by its Fourier modes

$$\Theta(\hat{x}) = \int \frac{d^3k}{(2\pi)^3}\Theta(\vec{k})e^{i\vec{k}\cdot\vec{x}}, \tag{2.3}$$

each Fourier mode corresponds to the mode of acoustic oscillations. The Gaussianity provides that Fourier components of the fluctuations are uncorrelated and have random phases.

Considering the $\hat{n}$-direction on the sky today, the temperature variation generated at the decoupling time is

$$\Theta(\hat{n}) = \int \frac{d^3k}{(2\pi)^3}\Theta(\vec{k})e^{i\vec{k}\cdot D_*\hat{n}}, \tag{2.4}$$

where $D_* = c(t_0 - t_*)$ is the distance traveled by light (CMB) from the decoupling time (t_*) to today (t_0).

One expands the plane wave in terms of the spherical harmonics

$$e^{i\vec{k}\cdot D_*\hat{n}} = 4\pi\sum_{lm} i^l j_l(kD_*)Y^*_{lm}(\hat{k})Y_{lm}(\hat{n}). \tag{2.5}$$

Substituting (2.1) and (2.5) in (2.4), one finds

$$\sum_{lm} a_{lm}Y_{lm}(\hat{n}) = 4\pi\int \frac{d^3k}{(2\pi)^3}\Theta(\vec{k})\sum_{lm} i^l j_l(kD_*)Y^*_{lm}(\hat{k})Y_{lm}(\hat{n}). \tag{2.6}$$

The orthogonality of the spherical harmonics provides the multipole moments as

$$a_{lm} = 4\pi\int \frac{d^3k}{(2\pi)^3}\Theta(\vec{k})i^l j_l(kD_*)Y_{lm}(\hat{k}). \tag{2.7}$$

By using the two-point correlation function,

$$< \Theta(\vec{k})^*\Theta(\vec{k}') >= (2\pi)^3\delta(\vec{k}-\vec{k}')P_T(k) \tag{2.8}$$

with the power spectrum $\mathcal{P}_T \equiv \frac{k^3}{2\pi^2}P_T(k)$, one computes $< a^*_{lm}a_{l'm'} >$ as

$$< a^*_{lm}a_{l'm'} >= \delta_{ll'}\delta_{mm'}4\pi\int d\ln k\, j^2_l(kD_*)\mathcal{P}^2_T(k) \tag{2.9}$$

For the slowly varying $\mathcal{P}^2_T(k)$ e.g. the power spectrum for the large scale perturbations, the dominant contribution comes from peaks of the spherical Bessel function. Since $j_l(x)$ is strongly peaked at $x \approx l$ (here $x = kD_*$), we can take $k = l/D_*$ as the characteristic scale corresponding to l [6]. From $\int_0^\infty j^2_l(x)d\ln x = [2l(l+1)]^{-1}$ the power spectrum is

$$\mathcal{P}^2_T(k)\Big|_{k\,=\,l/D_*} = \frac{l(l+1)}{2\pi}C_l. \tag{2.10}$$

The power spectrum of CMB anisotropies is plotted using the quantity $\frac{l(l+1)}{2\pi}C_l$ against the multipole l. Oscillation modes that reach an extrema of the oscillation become the peaks in the CMB power spectrum as shown in Figure 2.1.

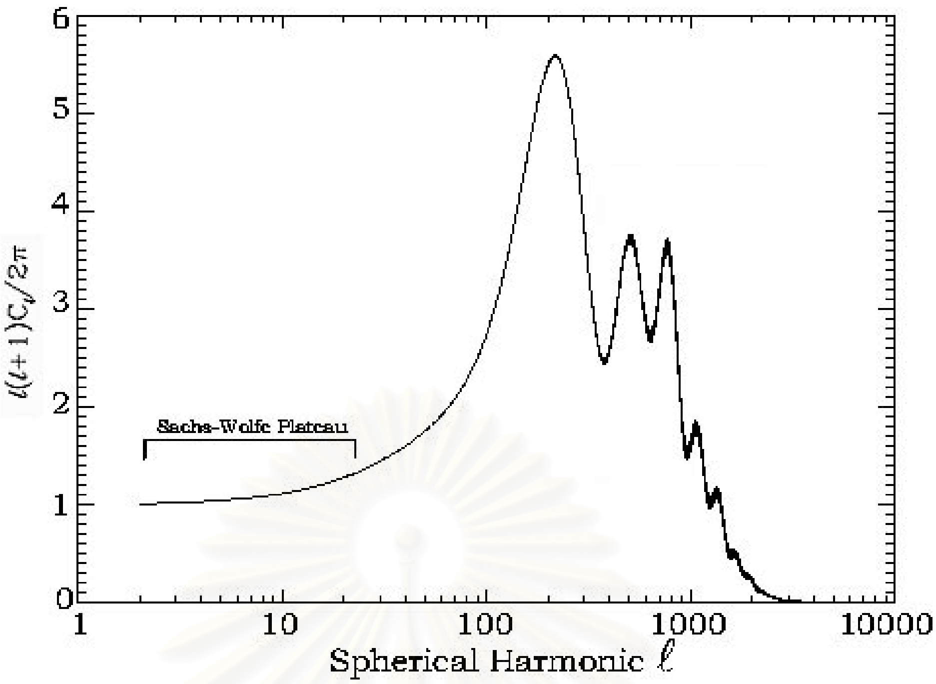

Figure 2.1: CMB power spectrum [10].

2.4.1 First Peak and Geometry of the Universe

The first peak corresponds to the sound horizon which is the distance the sound wave has propagated until the decoupling time, r_s. This mode is the longest wavelength of fluctuations of the acoustic oscillations. The fluctuations in this scale are the dominant fluctuations existed in the plasma at last scattering. Moreover, this mode reaches an extrema of the oscillation at decoupling, so the first peak is the highest peak in the power spectrum.

The location of the peak depends on the total matter in the universe, $l \sim \frac{200}{\sqrt{\Omega_{tot}}}$ (For flat universe: $\Omega_{tot} \simeq 1$). According to the power spectrum, one obtains $l \approx 200$, therefore we live in the flat space as the inflation predicted (another evidence is the data from WMAP: $\Omega_{tot} = 1.02 \pm 0.02$ [5]).

2.4.2 Relative Height of Peaks and Dark Matter

Another parameter that affects the height of every peak is the amount of baryons. If the amount of baryons is increased, the odd peaks corresponding to the compression of the plasma will be higher. This is because more baryons mean more mass which will cause more infall. On the other hand, the even peaks corresponding to the rarefaction of the plasma will be lower. The more baryons, the higher the odd peaks and the lower the even peaks.

All observations in [5] constrain the amount of the baryons in the universe to be $\Omega_b h^2 = 0.024 \pm 0.001$. If all matters contained in the universe are purely baryons: $\Omega_m h^2 \approx \Omega_b h^2$, the relative height of the first two peaks is less than that appears in Figure 2.1. In order to have the relative height between the odd and the even peaks as in Figure 2.1, the existence of the dark baryons or the dark matter is required. An inflation model called *double inflation*, which will be studied in Chapter V, can explain the origin of dark matter.

Although the acoustic peaks give much information about the universe, we are interested in the power spectrum on the angular scale greater than 10° (or $l \leq 20$) when we study the initial conditions or inflation.

2.5 Observational Data

As we know the sources of CMB temperature anisotropies come from the gravitation potential wells at the decoupling time, and the gravitation potential wells are caused by the quantum fluctuations during the inflation period. The two types of quantum fluctuations are combined into the curvature perturbation $\mathcal{R}$. Therefore from the equation (2.10),

$$\mathcal{P}^2_{\mathcal{R}}(k) = \mathcal{P}^2_T(k) = \frac{l(l+1)}{2\pi} C_l. \tag{2.11}$$

It can be seen that the CMB power spectrum is an important observation for testing inflation models. Moreover, the derivatives of the power spectrum are the other constraints for discriminating among inflation models. The WMAP data [5] on the large scale are

$$\begin{aligned} k &= 0.05\text{Mpc}^{-1} : \ \mathcal{P}_{\mathcal{R}} = 2.46 \times 10^{-9}, \ \ n_{\mathcal{R}} = 0.93 \pm 0.03, \ \ \frac{dn_{\mathcal{R}}}{d\ln k} = -0.031^{+0.016}_{-0.017}, \\ k &= 0.002\text{Mpc}^{-1} : \mathcal{P}_{\mathcal{R}} = 2.09 \times 10^{-9}, \ \ n_{\mathcal{R}} = 1.20^{+0.12}_{-0.11}, \ \ \frac{dn_{\mathcal{R}}}{d\ln k} = -0.077^{+0.050}_{-0.052}. \end{aligned}$$

where $n_{\mathcal{R}}$ and $\frac{dn_{\mathcal{R}}}{d\ln k}$ are the first and second derivatives of $\mathcal{P}_{\mathcal{R}}$ with respect to $\ln k$ respectively. Note that 1 Mpc $= 3.086 \times 10^{24}$ cm.

The scale $k = 0.05\text{Mpc}^{-1}$ corresponds to the galactic scale $L = 20$Mpc whereas the scale $k = 0.002\text{Mpc}^{-1}$ is for the cluster of galaxies $L = 500$Mpc. We use only the large scale observations because the temperature fluctuations on these scales are directly caused by the primordial perturbations in the early universe (inflation).

Another observation that can test inflation models is the CMB polarization [8, 34, 35]. The polarization of the CMB at the end of the decoupling time can give some information about the primordial perturbations. However, detecting the CMB polarization precisely is very difficult. In the near future, Planck satellite may give us the high-resolution map of the CMB polarization. In our study, we consider only the data of the CMB anisotropies.

CHAPTER III

INFLATION REVIEW

3.1 Scalar Field Dynamics

In this chapter, we consider a single scalar field, called *inflaton* during the period of inflation (the multi-field model will be considered in Chapter V).
The Lagrangian[1] of the inflaton, ϕ, is

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi), \tag{3.2}$$

where

$$g_{\mu\nu} = a^2(\eta) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.3}$$

is the the Friedmann-Robertson-Walker (FRW) metric in the unperturbed space-time with the *conformal* time coordinate, η. (For convenience, the proper time, t, is changed to the conformal time, η, by the definition $dt = a(\eta)d\eta$). And $a(\eta)$ is the scale factor which depends only on time. It tells us about the rate of the expansion of the universe.
The action for the inflation is

$$\begin{aligned} \mathcal{S} &= \int d^4x \sqrt{-g} \mathcal{L} && (3.4) \\ &= -\int d^4x \sqrt{-g} \Big[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \Big], && (3.5) \end{aligned}$$

[1] Normally, the Lagrangian is defined as

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi), \tag{3.1}$$

in this form, the metric is $g_{\mu\nu} = a^2 \eta_{\mu\nu}$ where $\eta_{\mu\nu} \equiv diag(1, -1, -1, -1)$ is the Minkowski metric. The metric $g_{\mu\nu}$ gives $\partial_\mu \phi \partial^\mu \phi = \left(\frac{\partial \phi}{\partial \eta}\right)^2 - (\nabla \phi)^2$. But for the notation above, $\partial_\mu \phi \partial^\mu \phi = -\left(\frac{\partial \phi}{\partial \eta}\right)^2 + (\nabla \phi)^2$ gives the opposite sign of the kinetic term.

where $\int d^4x\sqrt{-g}$ is the invariant volume element, and the determinant of the metric $g_{\mu\nu}$ is

$$g = \det g_{\mu\nu} = -a^8(\eta). \tag{3.6}$$

The invariance of the action under changing field configurations is

$$\begin{aligned}\delta\mathcal{S} &= 0 \\ &= -\int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\delta\phi\partial_\nu\phi + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\delta\phi + \delta V(\phi)\right] \\ &= -\int d^4x\left[\partial_\nu(g^{\mu\nu}\sqrt{-g}\partial_\mu\phi\delta\phi) - \Big(\partial_\nu(g^{\mu\nu}\sqrt{-g}\partial_\mu\phi) - \sqrt{-g}\frac{\delta V}{\delta\phi}\Big)\delta\phi\right].\end{aligned}$$

Due to vanishing variations at the boundary, the surface terms vanish. Then the equation of motion becomes

$$\frac{1}{\sqrt{-g}}\partial_\nu(g^{\mu\nu}\sqrt{-g}\partial_\mu\phi) - V_\phi = 0, \tag{3.7}$$

where V_ϕ is the derivative of the potential, V, with respect to the field, ϕ. Replacing $g_{\mu\nu}$ with $a^2(\eta)\eta_{\mu\nu}$, one gets the equation of motion for the inflaton in the conformal time coordinate [see appendix A.1]

$$\frac{\partial^2\phi}{\partial\eta^2} + \frac{2}{a}\big(\frac{da}{d\eta}\big)\frac{\partial\phi}{\partial\eta} - \nabla^2\phi + a^2V_\phi(\phi) = 0. \tag{3.8}$$

Spatially homogeneity implies that the gradient of ϕ vanishes, the equation of motion becomes

$$\frac{\partial^2\phi}{\partial\eta^2} + \frac{2}{a}\big(\frac{da}{d\eta}\big)\frac{\partial\phi}{\partial\eta} + a^2V_\phi(\phi) = 0. \tag{3.9}$$

The Hubble parameter in the conformal time and the proper time coordinates is

$$H = \frac{1}{a^2(\eta)}\frac{da}{d\eta} = \frac{\dot{a}}{a}. \tag{3.10}$$

From the definition, H, measures the rate of change of the scale factor, therefore one calls it the Hubble rate. The equation of motion in terms of the Hubble parameter can be written as follow,

$$\frac{\partial^2\phi}{\partial\eta^2} + 2aH\frac{\partial\phi}{\partial\eta} + a^2V_\phi(\phi) = 0. \tag{3.11}$$

The energy-momentum tensor for the inflation is given by

$$\begin{aligned}T_{\mu\nu} &= \partial_\mu\phi\partial_\nu\phi + \mathcal{L}g_{\mu\nu} \\ &= \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\big(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\big)\end{aligned} \tag{3.12}$$

For any perfect fluid which has no viscosity and heat flow. The energy-momentum tensor is

$$T^{\mu}{}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{3.13}$$

Here ρ and p are the density and the pressure of the perfect fluid respectively. Because $T_{\mu\nu} = g_{\mu\beta}T^{\beta}{}_{\nu}$, one obtains

$$T_{\mu\nu} = a^2(\eta) \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{3.14}$$

Considering the inflaton field as a homogeneous perfect fluid ($\nabla\phi = 0$), its energy-momentum tensor in (3.12) can be written in components as

$$T_{00} = \Big[\frac{1}{2}\big(\frac{\partial\phi}{\partial\eta}\big)^2 + V(\phi)a^2(\eta)\Big], \tag{3.15a}$$

$$T_{0i} = 0, \tag{3.15b}$$

$$T_{ij} = \Big[\frac{1}{2}\big(\frac{\partial\phi}{\partial\eta}\big)^2 - V(\phi)a^2(\eta)\Big]\delta_{ij}. \tag{3.15c}$$

Comparing the results with (3.14), the energy density and the pressure of the inflaton are

$$\rho = \frac{1}{2a^2(\eta)}\big(\frac{\partial\phi}{\partial\eta}\big)^2 + V(\phi), \tag{3.16a}$$

$$p = \frac{1}{2a^2(\eta)}\big(\frac{\partial\phi}{\partial\eta}\big)^2 - V(\phi). \tag{3.16b}$$

It can be seen that when the potential energy of the inflaton is larger than its kinetic energy, the negative pressure appears. This condition is very important in order to have inflation ($\ddot{a} > 0$).
The continuity equation derived from $\nabla_{\mu}T^{\mu}{}_{0} = 0$ is

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{3.17}$$

The equation of state is

$$p = \omega\rho, \tag{3.18}$$

where ω is a number depending on fluid types. The Friedmann equations derived by Einstein equation are

$$\frac{1}{a^2(\eta)}\Big(\frac{da}{d\eta}\Big)^2 + \mathrm{k}^2 = \frac{\rho a^2}{3m_{pl}^2}, \tag{3.19}$$

$$\frac{1}{a(\eta)}\Big(\frac{d^2a}{d\eta^2}\Big) - \frac{1}{a^2(\eta)}\Big(\frac{da}{d\eta}\Big)^2 = -\frac{a^2}{6m_{pl}^2}(\rho+3p), \tag{3.20}$$

where $m_{pl} \equiv \frac{1}{\sqrt{8\pi G}}$ is the reduced Planck mass, and k is the curvature constant describing the geometry of the universe in three-dimensional space.
Using (3.16), the Friedmann equations in the flat universe ($\mathrm{k} = 0$) become

$$\frac{1}{a^2(\eta)}\Big(\frac{da}{d\eta}\Big)^2 = \frac{1}{3m_{pl}^2}\Big[a^2(\eta)V(\phi) + \frac{1}{2}\Big(\frac{\partial\phi}{\partial\eta}\Big)^2\Big], \tag{3.21}$$

$$\frac{1}{a(\eta)}\Big(\frac{d^2a}{d\eta^2}\Big) - \frac{1}{a^2(\eta)}\Big(\frac{da}{d\eta}\Big)^2 = \frac{1}{3m_{pl}^2}\Big[a^2(\eta)V(\phi) - \Big(\frac{\partial\phi}{\partial\eta}\Big)^2\Big]. \tag{3.22}$$

Using the equation of state, the continuity equation and the (flat) Friedmann equations, one finds

$$a \propto \rho^{3(1+\omega)}, \quad a \propto t^{\frac{2}{3(1+\omega)}}, \tag{3.23}$$

this shows that the energy density of the universe determines its evolution[2]. For the negative pressure in the inflation epoch ($\omega < 0$): $a \propto t^p$, $p > 1$. Notice that, the distance traveled by light is $a = ct,\ c = 1$. Therefore during inflation epoch, the spacetime expands faster than the speed of light.

In the next section we will consider all conditions necessary for having inflation and the relevant parameters are included there.

3.2 Slow Roll Conditions

From (3.9), the field equation in the proper time coordinate, t, can be written as

$$\ddot{\phi} + 3\Big(\frac{\dot{a}}{a}\Big)\dot{\phi} + V_\phi(\phi) = 0. \tag{3.24}$$

In the inflation period, the energy density of the universe is dominated by the inflaton potential energy. It means $V(\phi) \gg \dot{\phi}^2$. Moreover, the friction is large, so the inflaton rolls slowly along the potential with a constant velocity (as in the

[2]For example, during the epoch of relativistic matter ($\omega = \frac{1}{3}$): $a \propto \rho^4$ and $a \propto t^{\frac{1}{2}}$. During the epoch of non-relativistic matter ($\omega = 0$): $a \propto \rho^3$ and $a \propto t^{\frac{2}{3}}$.

case of a ball falls in a fluid with high viscosity). Then the $\ddot{\phi}$-term in the equation of motion can be neglected.

$$3H\dot{\phi} + V_{\phi}(\phi) \simeq 0. \tag{3.25}$$

The first Friedmann equation (in the proper time coordinate) in the slow roll limit, $\frac{(\dot{\phi})^2}{2} \ll V(\phi)$, is

$$H^2 \simeq \frac{V(\phi)}{3m_{pl}^2}. \tag{3.26}$$

Using (3.25) and (3.26), one obtains

$$\begin{aligned} \frac{(\dot{\phi})^2}{2} &= \frac{m_{pl}^2}{6}\frac{[V_{\phi}(\phi)]^2}{V(\phi)} \\ &\ll V(\phi), \end{aligned} \tag{3.27}$$

therefore the first consistency condition is:

$$\frac{m_{pl}^2}{6}\left[\frac{V_{\phi}(\phi)}{V(\phi)}\right]^2 \ll 1, \tag{3.28}$$

$$\left|m_{pl}\frac{V_{\phi}(\phi)}{V(\phi)}\right| \ll \sqrt{6}. \tag{3.29}$$

Differentiating (3.27) with respect to time, one has

$$\begin{aligned} \frac{1}{2}(2\dot{\phi})\ddot{\phi} &= \frac{m_{pl}^2}{6}\left[\frac{2V_{\phi}(\phi)V_{\phi\phi}(\phi)\dot{\phi}}{V(\phi)} - \left[\frac{V_{\phi}(\phi)}{V(\phi)}\right]^2 V_{\phi}(\phi)\dot{\phi}\right] \\ \ddot{\phi} &= \frac{m_{pl}^2}{3}\frac{V_{\phi}(\phi)V_{\phi\phi}(\phi)}{V(\phi)} - \frac{m_{pl}^2}{6}\left[\frac{V_{\phi}(\phi)}{V(\phi)}\right]^2 V_{\phi}(\phi). \end{aligned} \tag{3.30}$$

The last term is neglected due to the first consistency condition in (3.28). Besides, the result from the slow roll approximation in (3.25) gives $\ddot{\phi} \ll V_{\phi}(\phi)$. Then

$$\begin{aligned} |\ddot{\phi}| &\simeq \left|\frac{m_{pl}^2}{3}\frac{V_{\phi}(\phi)V_{\phi\phi}(\phi)}{V(\phi)}\right| \ll |V_{\phi}(\phi)|, \\ \left|\frac{m_{pl}^2}{3}\frac{V_{\phi\phi}(\phi)}{V(\phi)}\right| &\ll 1, \end{aligned}$$

or

$$\left|m_{pl}^2\frac{V_{\phi\phi}(\phi)}{V(\phi)}\right| \ll 3. \tag{3.31}$$

This is the second consistency condition.

These two slow roll consistency conditions require a very flat inflaton potential.

This kind of potential also gives the large amount of inflation or the large amount of e-folding, N, which is defined by[3]

$$N = \int_{t_i}^{t_f} H(t)dt, \tag{3.32}$$

where t_i and t_f are the time at the beginning and the end of the inflation period. From (3.26), the number of e-folding is [see appendix A.2]

$$N = \int_{\phi_f}^{\phi_i} \frac{1}{m_{pl}^2}\frac{V(\phi)}{V_\phi(\phi)}d\phi. \tag{3.33}$$

The Friedmann equation in (3.21) can be written in terms of the Hubble parameter as

$$H^2 = \tfrac{1}{a^4(\eta)}(\tfrac{da}{d\eta})^2 = \tfrac{1}{3m_{pl}^2}\left[V(\phi) + \tfrac{1}{a^2(\eta)}(\tfrac{\partial\phi}{\partial\eta})^2\right]. \tag{3.34}$$

Differentiating the equation with respect to the conformal time, one finds

$$\frac{dH}{d\eta} = -\frac{1}{2m_{pl}^2 a(\eta)}\Big(\frac{d\phi}{d\eta}\Big)^2, \tag{3.35}$$

$$H_\phi(\phi) = -\frac{1}{2m_{pl}^2 a(\eta)}\frac{d\phi}{d\eta}. \tag{3.36}$$

Here, $H_\phi \equiv \frac{\partial H}{\partial\phi}$. Using (3.34) and (3.36), we have

$$V(\phi) = 2m_{pl}^4\left[\frac{3}{2m_{pl}^2}H^2(\phi) - \big(H_\phi(\phi)\big)^2\right] \tag{3.37}$$

$$= 3m_{pl}^2 H^2(\phi)\left[1 - \frac{2m_{pl}^2}{3}\left[\frac{H_\phi(\phi)}{H(\phi)}\right]^2\right]$$

$$\equiv 3m_{pl}^2 H^2(\phi)\Big[1 - \frac{1}{3}\epsilon(\phi)\Big], \tag{3.38}$$

where ϵ is a slow roll parameter defined by[4]

$$\epsilon(\phi) \equiv 2m_{pl}^2\Big[\frac{H_\phi(\phi)}{H(\phi)}\Big]^2. \tag{3.39}$$

The derivatives of the potential $V_\phi(\phi)$ and $V_{\phi\phi}(\phi)$ are [see appendix A.3]

$$V_\phi(\phi) = -3\sqrt{2}m_{pl}H^2(\phi)\sqrt{\epsilon(\phi)}\big[1 + \delta_1(\phi)\big], \tag{3.40}$$

$$V_{\phi\phi}(\phi) = 3H^2\Big[\epsilon(\phi) - \delta_1(\phi) - \frac{1}{3}\delta_1^2(\phi) - \frac{1}{3}\dot{\phi}\delta_2(\phi)\Big], \tag{3.41}$$

[3] $N(t) = \int_t^{t_f} H(t')dt'$ is the e-folds between a particular time, t, and the end of inflation.

[4] $\epsilon = 2m_{pl}^2\left[\frac{H_\phi(\phi)}{H(\phi)}\right]^2 = -\frac{\dot{H}}{H^2} = \frac{1}{2m_{pl}^2}\left(\frac{\dot{\phi}}{H}\right)^2$.

where the other slow roll parameters are defined by

$$\delta_n \equiv \frac{1}{H^n \dot{\phi}} \frac{d^{n+1}\phi}{dt^{n+1}}. \tag{3.42}$$

The Friedmann equation in (3.22) is rewritten in term of the slow roll parameter, ϵ, as [see appendix A.4]

$$\left(\frac{\ddot{a}}{a}\right) = H^2(\phi)\big[1 - \epsilon(\phi)\big]. \tag{3.43}$$

It can be seen that $\epsilon < 1$ gives $\ddot{a} > 0$ which is a condition for inflation. Moreover, the large amount of inflation is needed in order that quantum fluctuations are enlarged to be the large scale structure today. It requires $\epsilon \ll 1$ as well as $\delta_n \ll 1$. From the equations (3.40) and (3.41), the limit of the slow roll parameters gives $V_\phi(\phi) \ll 1$ and $V_{\phi\phi}(\phi) \ll 1$ which satisfy the consistency conditions.

The equation of state in the inflation period can be expressed as [see appendix A.5]

$$p = -\rho\Big(1 - \frac{2}{3}\epsilon(\phi)\Big). \tag{3.44}$$

It shows that $-\rho \le p < -\frac{\rho}{3}$ or $-1 \le \omega < -\frac{1}{3}$ during inflation.

Now we have already known about the slow roll conditions for inflaton field dynamics. Next section, using the conditions, dynamics of inflaton field fluctuations in the perturbed spacetime are studied.

3.3 Perturbed Einstein Equation

The two sections above concern the dynamics of the homogeneous inflaton field in the homogeneous spacetime. But the universe cannot be perfectly homogeneous in the past because of inhomogeneities today. In this section all fluctuations are considered as perturbations in the Einstein equation.

3.3.1 Metric Fluctuations

The most generic perturbed FRW-metric is written as [11]

$$\begin{aligned} ds^2 &= a^2\big((-1 - 2\,A)d\tau^2 + 2\,B_{,i}\; d\tau\, dx^i + (1 - 2\,\psi)\delta_{ij}dx^i\, dx^j\big) \\ &\quad + a^2\big(E_{,ij}\; dx^i\, dx^j\big). \end{aligned} \tag{3.45}$$

where $B_{,i} \equiv \frac{\partial B}{\partial x^i}$ and $E_{,ij} \equiv \frac{\partial E}{\partial x^i \partial x^j}$. One transforms the coordinates as follow

$$\begin{aligned} \eta &\rightarrow \eta - (B - E') \\ x^i &\rightarrow x^i + \gamma^{ij} E_{,j} \end{aligned} \tag{3.46}$$

(γ^{ij} is the spatial metric) to simplify the problem. The perturbed FRW-metric becomes

$$ds^2 = a^2\big(-(1 + 2\,A) d\eta^2 + ((1 - 2\,\psi)\delta_{ij}\,)\; dx^i\, dx^j \big). \tag{3.47}$$

This is called the *longitudinal gauge* which keeps only the scalar perturbations. From (3.47), the metric tensor is

$$\mathrm{g}_{\mu\nu} = a^2 \begin{pmatrix} -1 - 2\,A & 0 \\ 0 & (1 - 2\,\psi)\,\delta_{ij} \end{pmatrix}, \tag{3.48}$$

$$= g_{\mu\nu} + \delta g_{\mu\nu}. \tag{3.49}$$

Here the unperturbed metric is

$$g_{\mu\nu} = a^2 \begin{pmatrix} -1 & 0 \\ 0 & \delta_{ij} \end{pmatrix}, \tag{3.50}$$

while the perturbed metric is

$$\delta g_{\mu\nu} = a^2 \begin{pmatrix} -2A & 0 \\ 0 & 2\psi\delta_{ij} \end{pmatrix}. \tag{3.51}$$

From $g^{\mu\alpha}\, g_{\alpha\nu} = \delta^{\mu}{}_{\nu}$, the lowest order of the unperturbed inverse metric is

$$g^{\mu\nu} = \frac{1}{a^2} \begin{pmatrix} -1 & 0 \\ 0 & \delta^{ij} \end{pmatrix}, \tag{3.52}$$

so the inverse metric of $\mathrm{g}_{\mu\nu}$ can be written in general,

$$\mathrm{g}^{\mu\nu} = a^2 \begin{pmatrix} -1 + X & 0 \\ 0 & (1 + Y)\,\delta^{ij} \end{pmatrix}. \tag{3.53}$$

The 00-component is

$$\begin{aligned} \mathrm{g}^{0\alpha}\, \mathrm{g}_{\alpha 0} &= -(-1 + X)(1 + 2\,A) = 1, \\ X &= 2A. \end{aligned} \tag{3.54}$$

The ij-component is

$$\begin{aligned} \mathrm{g}^{i\alpha}\, \mathrm{g}_{\alpha j} &= (1 + Y)\,\delta^{il}\,(1 - 2\,\psi)\,\delta_{lj} = \delta^{i}{}_{j}, \\ Y &= 2\psi. \end{aligned} \tag{3.55}$$

So, the inverse metric is

$$\mathrm{g}^{\mu\nu} = \frac{1}{a^2}\begin{pmatrix} -1 + 2A & 0 \\ 0 & (1 + 2\psi)\,\delta^{ij} \end{pmatrix}. \tag{3.56}$$

where

$$\delta g^{\mu\nu} = \frac{2}{a^2}\begin{pmatrix} A & 0 \\ 0 & \psi\,\delta^{ij} \end{pmatrix}. \tag{3.57}$$

The perturbations in the spacetime metric, $\delta g_{\mu\nu}$, lead to the perturbations in the Christoffel connections which will be studied in the next subsection.

3.3.2 Perturbed Christoffel Connections

The Christoffel connections, $\Gamma^{\alpha}_{\beta\gamma}$, are defined in terms of the metric tensor as,

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}\, g^{\alpha\rho}\,(g_{\rho\gamma,\beta} + g_{\beta\rho,\gamma} - g_{\beta\gamma,\rho})\,, \tag{3.58}$$

where $g_{\rho\gamma,\beta} \equiv \frac{\partial g_{\rho\gamma}}{\partial x^{\beta}}$. The first order in the perturbations of the connections are

$$\begin{aligned}\delta\Gamma^{\alpha}_{\beta\gamma} &= \frac{1}{2}\,\delta g^{\alpha\rho}\,(g_{\rho\gamma,\beta} + g_{\beta\rho,\gamma} - g_{\beta\gamma,\rho}) \\ &\quad + \frac{1}{2}\, g^{\alpha\rho}\,(\delta g_{\rho\gamma,\beta} + \delta g_{\beta\rho,\gamma} - \delta g_{\beta\gamma,\rho})\,. \end{aligned} \tag{3.59}$$

By the computation in appendix B.1, the non-zero components of the unperturbed Christoffel connections are

$$\Gamma^{0}_{00} = \frac{a'}{a}\,; \qquad \Gamma^{i}_{0j} = \frac{a'}{a}\,\delta^{i}{}_{j}\,; \qquad \Gamma^{0}_{ij} = \frac{a'}{a}\,\delta_{ij}, \tag{3.60}$$

where a' is the derivative of a with respect to the conformal time, η.
The perturbed part of the connections are

$$\delta\Gamma^{0}_{00} = A'\,; \tag{3.61}$$

$$\delta\Gamma^{0}_{0i} = \partial_i\, A\,; \tag{3.62}$$

$$\delta\Gamma^{i}_{00} = \partial^i A\,; \tag{3.63}$$

$$\delta\Gamma^{0}_{ij} = -\,2\,\frac{a'}{a}\,(A + \psi)\,\delta_{ij} \,-\, \psi'\,\delta_{ij}\,; \tag{3.64}$$

$$\delta\Gamma^{i}_{0j} = -\,\psi'\delta^{i}{}_{j}\,; \tag{3.65}$$

$$\delta\Gamma^{i}_{jk} = -\,\partial_j\psi\,\delta_k{}^{i} \,-\, \partial_k\psi\,\delta_j{}^{i} \,+\, \partial^i\psi\,\delta_{jk}\,. \tag{3.66}$$

3.3.3 Perturbed Ricci Tensor

In order to compute the perturbed Einstein equation, the perturbed Ricci tensor and Ricci scalar are needed.
The Ricci tensor is defined by

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\nu\alpha} + \Gamma^\alpha_{\sigma\alpha}\Gamma^\sigma_{\mu\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\mu\alpha}. \tag{3.67}$$

The first order in the perturbations of the tensor is

$$\begin{aligned} \delta R_{\mu\nu} &= \partial_\alpha \delta\Gamma^\alpha_{\mu\nu} - \partial_\mu \delta\Gamma^\alpha_{\nu\alpha} + \delta\Gamma^\alpha_{\sigma\alpha}\Gamma^\sigma_{\mu\nu} + \Gamma^\alpha_{\sigma\alpha}\delta\Gamma^\sigma_{\mu\nu} \\ &\quad - \delta\Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\mu\alpha} - \Gamma^\alpha_{\sigma\nu}\delta\Gamma^\sigma_{\mu\alpha}. \end{aligned} \tag{3.68}$$

The non-zero components of the unperturbed Ricci tensor are [see appendix B.2]

$$R_{00} = -3\frac{a''}{a} + 3\left(\frac{a'}{a}\right)^2; \tag{3.69}$$

$$R_{ij} = \left(\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right)\delta_{ij}, \tag{3.70}$$

whereas the components of the perturbed part are

$$\delta R_{00} = \partial_i\partial^i A + 3\psi'' + 3\frac{a'}{a}\psi' + 3\frac{a'}{a}A'; \tag{3.71}$$

$$\delta R_{0i} = 2\partial_i\psi' + 2\frac{a'}{a}\partial_i A; \tag{3.72}$$

$$\begin{aligned} \delta R_{ij} &= \Big(-\frac{a'}{a}A' - 5\frac{a'}{a}\psi' - 2\frac{a''}{a}A - 2\left(\frac{a'}{a}\right)^2 A - 2\frac{a''}{a}\psi \\ &\quad -2\left(\frac{a'}{a}\right)^2\psi - \psi'' + \partial_k\partial^k\psi\Big)\delta_{ij} + \partial_i\partial_j\psi - \partial_i\partial_j A. \end{aligned} \tag{3.73}$$

3.3.4 Perturbed Ricci Scalar

The Ricci scalar is defined by contracting the Ricci tensor with the metric tensor

$$R = g^{\mu\alpha}R_{\alpha\mu}, \tag{3.74}$$

Its first order perturbations are

$$\delta R = \delta g^{\mu\alpha}R_{\alpha\mu} + g^{\mu\alpha}\delta R_{\alpha\mu}. \tag{3.75}$$

Its unperturbed part is [see appendix B.3]

$$R = \frac{6}{a^2}\frac{a''}{a}, \tag{3.76}$$

while the perturbed part is

$$\delta R = -\frac{1}{a^2}\Big(2\partial_i\partial^i A + 6\psi'' + 6\frac{a'}{a}A' + 18\frac{a'}{a}\psi' + 12\frac{a''}{a}A - 4\partial_i\partial^i\psi\Big). \tag{3.77}$$

Now the perturbed Einstein tensor can be derived.

3.3.5 Perturbed Einstein Tensor

The Einstein tensor is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{3.78}$$

The first order perturbations of the Einstein tensor can be written in terms of the perturbed Ricci tensor and Ricci scalar as follow

$$\delta G_{\mu\nu} = \delta R_{\mu\nu} - \frac{1}{2} \delta g_{\mu\nu} R - \frac{1}{2} g_{\mu\nu} \delta R. \tag{3.79}$$

The non-zero unperturbed components of the tensor are [see appendix B.4]

$$G_{00} = 3 \left(\frac{a'}{a}\right)^2 ; \tag{3.80}$$

$$G_{ij} = \left(-2\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right) \delta_{ij} . \tag{3.81}$$

The perturbed components are

$$\delta G_{00} = -6\frac{a'}{a} \psi' + 2 \partial_i \partial^i \psi ; \tag{3.82}$$

$$\delta G_{0i} = 2\partial_i \psi' + 2 \frac{a'}{a} \partial_i A ; \tag{3.83}$$

$$\begin{aligned} \delta G_{ij} = & \left(2\frac{a'}{a} A' + 4\frac{a'}{a}\psi' + 4\frac{a''}{a} A - 2\left(\frac{a'}{a}\right)^2 A + 4\frac{a''}{a}\psi \right. \\ & \left. -2\left(\frac{a'}{a}\right)^2 \psi + 2\psi'' - \partial_k\partial^k \psi + \partial_k \partial^k A\right)\delta_{ij} \\ & + \partial_i\partial_j\psi - \partial_i\partial_j A. \end{aligned} \tag{3.84}$$

From $G^{\mu}{}_{\nu} = g^{\mu\alpha} G_{\alpha\nu}$, so

$$\begin{aligned} \delta G^{\mu}{}_{\nu} &= \delta(g^{\mu\alpha} G_{\alpha\nu}) \\ &= \delta g^{\mu\alpha} G_{\alpha\nu} + g^{\mu\alpha} \delta G_{\alpha\nu}. \end{aligned} \tag{3.85}$$

The non-zero components of the Einstein tensor are

$$\delta G^{0}{}_{0} = \frac{1}{a^2}\left(6\left(\frac{a'}{a}\right)^2 A + 6\frac{a'}{a}\psi' - 2\partial_i\partial^i\psi\right); \tag{3.86}$$

$$\delta G^{0}{}_{i} = \frac{1}{a^2}\left(-2\partial_i\psi' - 2\frac{a'}{a}\partial_i A\right); \tag{3.87}$$

$$\begin{aligned} \delta G^{i}{}_{j} = \frac{1}{a^2}\Bigg\{ & \left(2\frac{a'}{a}A' + 4\frac{a''}{a}A - 2\left(\frac{a'}{a}\right)^2 A + \partial_k\partial^k A + 4\frac{a'}{a}\psi' \right. \\ & \left. + 2\psi'' - \partial_k\partial^k\psi\right)\delta^{i}{}_{j} - \partial^i\partial_j A + \partial^i\partial_j\psi \Bigg\}. \end{aligned} \tag{3.88}$$

Raising one of the indices of the Einstein tensor makes the computation shorter as we will see later.

We have already obtained the geometrical perturbations in form of the perturbed Einstein tensor. Now the perturbations of the energy momentum tensor in needed in order to complete the Einstein equation.

3.3.6 Perturbed Energy-Momentum Tensor

In this subsection, the matter part in the Einstein equation is considered. Matter is represented in the energy-momentum tensor whose background part is

$$T_{\mu\nu} \,=\, \partial_\mu\phi\,\partial_\nu\phi \,-\, g_{\mu\nu}\left(\frac{1}{2}\,g^{\alpha\beta}\,\partial_\alpha\phi\,\partial_\beta\phi \,+\, V(\phi)\right), \tag{3.89}$$

while its perturbed part is

$$\begin{aligned}
\delta T_{\mu\nu} \quad &= \quad \partial_\mu\delta\phi\,\partial_\nu\phi \,+\, \partial_\mu\phi\,\partial_\nu\delta\phi \,-\, \delta g_{\mu\nu}\left(\frac{1}{2}\,g^{\alpha\beta}\,\partial_\alpha\phi\,\partial_\beta\phi \,+\, V(\phi)\right) \\
&\quad - g_{\mu\nu}\left(\frac{1}{2}\delta g^{\alpha\beta}\,\partial_\alpha\phi\,\partial_\beta\phi + g^{\alpha\beta}\,\partial_\alpha\delta\phi\,\partial_\beta\phi + V_\phi\,\delta\phi\right).
\end{aligned} \tag{3.90}$$

The background components are the same as in (3.15)

$$\begin{aligned}
T_{00} &= \frac{1}{2}\,\phi'^2 \,+\, V(\phi)\,a^2\,; \\
T_{0i} &= 0\,; \\
T_{ij} &= \left(\frac{1}{2}\,\phi'^2 \,-\, V(\phi)\,a^2\right)\delta_{ij}\,,
\end{aligned} \tag{3.91}$$

whereas the perturbed components are [see appendix B.5]

$$\delta T_{00} \;=\; \delta\phi'\,\phi' \,+\, 2\,A\,V(\phi)\,a^2 \,+\, a^2\,V_\phi\,\delta\phi\,; \tag{3.92}$$

$$\delta T_{0i} \;=\; \partial_i\,\delta\phi\,\phi'\,; \tag{3.93}$$

$$\delta T_{ij} \;=\; \left(\delta\phi'\,\phi' - A\,\phi'^2 - a^2\,V_\phi\,\delta\phi - \psi\,\phi'^2 + 2\,\psi\,V(\phi)\,a^2\right)\delta_{ij}\,. \tag{3.94}$$

Again,

$$\begin{aligned}
\delta T^{\mu}{}_{\nu} &= \delta(g^{\mu\alpha}\,T_{\alpha\nu}) \\
&= \delta g^{\mu\alpha}\,T_{\alpha\nu} \,+\, g^{\mu\alpha}\,\delta T_{\alpha\nu},
\end{aligned} \tag{3.95}$$

which is written in components as

$$\delta T^{0}{}_{0} \;=\; \frac{1}{a^2}\left(A\,\phi'^2 \,-\, \delta\phi'\,\phi' \,-\, \delta\phi\,\frac{\partial V}{\partial\phi}\,a^2\right)\,; \tag{3.96}$$

$$\begin{aligned}
\delta T^{0}{}_{i} &= \frac{1}{a^2}\left(-\,\partial_i\delta\phi\,\phi'\right)\,; \\
\delta T^{i}{}_{j} &= \frac{1}{a^2}\left(-\,A\,\phi'^2 \,+\, \delta\phi'\,\phi' \,-\, \delta\phi\,\frac{\partial V}{\partial\phi}\,a^2\right)\delta^{i}{}_{j}\,.
\end{aligned} \tag{3.97}$$

We are ready to find the Einstein equation in each component.

3.3.7 Perturbed Einstein Equation

The background Einstein equations:

$$\begin{aligned} G_{00} &= \frac{1}{m_{pl}^2} T_{00} \\ \left(\frac{a'}{a}\right)^2 &= \frac{1}{3m_{pl}^2}\left(\frac{1}{2}\,\phi'^2 + V(\phi)\,a^2\right). \end{aligned} \tag{3.98}$$

The equation above is the first Friedmann equation while the second can be derived from the ij-component of the Einstein equation below

$$\begin{aligned} G_{ij} &= \frac{1}{m_{pl}^2} T_{ij} \\ \frac{a''}{a} - \left(\frac{a'}{a}\right)^2 &= \frac{1}{3m_{pl}^2}\left(V(\phi)\,a^2 - \phi'^2\right). \end{aligned} \tag{3.99}$$

In the proper time coordinate, the second Friedmann equation is

$$\frac{\ddot{a}}{a} = \frac{1}{3m_{pl}^2}\left(V(\phi) - \frac{1}{a^2}\phi'^2\right), \tag{3.100}$$

where $\ddot{a} = \frac{1}{a}\left[\frac{a''}{a} - \left(\frac{a'}{a}\right)^2\right]$.

The perturbed Einstein equation is considered. Since there are no non-diagonal component in the energy momentum tensor, the non-diagonal part of the Einstein tensor is equal to zero [5]

$$\begin{aligned} \partial^i\partial_j\psi - \partial^i\partial_j A &= 0 \\ \psi &= A. \end{aligned} \tag{3.101}$$

The components of the perturbed Einstein equation are [see appendix B.6]

$$3\mathcal{H}^2\psi + 3\mathcal{H}\psi' - \nabla^2\psi = \frac{1}{2m_{pl}^2}\left(\psi\phi'^2 - \delta\phi'\phi' - \delta\phi V_\phi a^2\right); \tag{3.102}$$

$$\mathcal{H}\psi + \psi' = \frac{1}{2m_{pl}^2}\left(\delta\phi\phi'\right); \tag{3.103}$$

$$\mathcal{H}^2\psi + 2\mathcal{H}'\psi + 3\mathcal{H}\psi' + \psi'' = \frac{1}{2m_{pl}^2}\left(-\psi\phi'^2 + \delta\phi'\phi' - \delta\phi V_\phi a^2\right), \tag{3.104}$$

---

[5] Consider Bardeen's potentials [11]

$$\begin{aligned} \Phi &= -A + \frac{1}{a}\left[\left(-B + \frac{E'}{2}\right)a\right]', \\ \Psi &= -\psi - \frac{1}{6}\nabla^2 E + \frac{a'}{a}\left(B - \frac{E'}{2}\right), \end{aligned}$$

which are gauge invariant. For longitudinal gauge, one finds

$$\Phi = -A, \qquad \Psi = -\psi.$$

where $\mathcal{H} \equiv a'/a = aH$ and $a''/a = \mathcal{H}^2 + \mathcal{H}'$.
Summing (3.102) and (3.104), replacing $\delta\phi$ by using (3.103), then using the (homogeneous) field equation, one obtains

$$\psi'' + 2\left(\mathcal{H} - \frac{\phi''}{\phi'}\right)\psi' - \nabla^2\psi + 2\left(\mathcal{H}' - \mathcal{H}\frac{\phi''}{\phi'}\right)\psi = 0. \quad (3.105)$$

From the perturbed Einstein equation, the perturbation in the matter field, $\delta\phi(\eta, \vec{x})$, leads to the perturbation in the spacetime metric, $\psi(\eta, \vec{x})$, and vise versa. After the inflation era, the two quantum perturbations, $\delta\phi$ and ψ, are enlarged to be the classical perturbations called *primordial perturbations*. Their solutions in the long wavelength limit and short wavelength limit will be considered in the next section.

3.4 Primordial Perturbations and Their Power Spectrum

During the period of inflation, spacetime expands faster than the speed of light. So the wavelengths of perturbations are stretched outside the causal-contact region called *horizon*.[6] Their magnitudes are assumed to be constant if there are no other sources (entropy perturbations) when they leave outside the horizon. However, the metric perturbation, ψ, is not constant even in the large scale [12] ($\lambda > \frac{H^{-1}}{a}$ or $k < aH$ where k is the comoving wave number), one introduces the comoving curvature perturbation, $\mathcal{R}$, which takes the form

$$\mathcal{R} \equiv \psi + \frac{\mathcal{H}}{\phi'}\delta\phi. \quad (3.106)$$

This is for a single field inflation model, which has no entropy perturbation,[7] thus the curvature perturbation is constant (will be proved later) in the large scale.

---

[6]the furthest distance/area that light can travel: the furthest distance that the observer can observe. The region inside the horizon contains all events that are visible while the outside contains all events that cannot be observed by the observer. From the definition of the Hubble parameter, the comoving horizon size is $\frac{H^{-1}}{a}$.

[7]There are no entropy perturbation in single field models, therefore perturbations are purely adiabatic (will be explained in Chapter V).

Substituting (3.106) in (3.105), the result is[8]

$$\mathcal{R}'' + 2\frac{z'}{z^2}\mathcal{R}' - \nabla^2\mathcal{R} = 0. \tag{3.108}$$

Here, $z \equiv a\phi'/\mathcal{H} = a\dot{\phi}/H$.

In order to eliminate the friction term, the curvature perturbation is replaced by a gauge invariant quantity, $u = z\mathcal{R}$. The equation of motion for $u(\eta, \vec{x})$ is

$$u'' - \nabla^2 u - \frac{z''}{z}u = 0. \tag{3.109}$$

Inflation occurs when the size of the universe is about the Planck scale in which quantum effect cannot be neglected. We quantize the perturbation field by expanding it into a creation operator, $\hat{a}^\dagger_{\vec{k}}$, and an annihilation operator, $\hat{a}_{\vec{k}}$.

$$u(\eta, \vec{x}) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \left(u_k(\eta)\hat{a}_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} + u_k^*(\eta)\hat{a}^\dagger_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} \right) \tag{3.110}$$

where the two operators satisfy the commutator $[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}] = \delta^{(3)}(\vec{k} - \vec{k'})$.

The equation of motion becomes

$$u_k'' + \left(k^2 - \frac{z''}{z} \right) u_k = 0. \tag{3.111}$$

Computing z''/z in terms of the slow roll parameters (appendix C.1), the result is

$$\frac{z''}{z} = 2(aH)^2 \left(1 + \epsilon + \frac{3}{2}\delta_1 + \frac{\delta_2}{2} + \epsilon^2 + 2\epsilon\delta_1 \right). \tag{3.112}$$

Recall that the slow roll parameters are small because of the almost flat inflaton potential, one needs to keep the first few orders in the slow roll parameters. Using

$$aH = \frac{-1}{\eta(1-\epsilon)}, \quad -\infty < \eta < 0, \tag{3.113}$$

one finds

$$\begin{aligned} \frac{z''}{z} &= \frac{2}{\eta^2(1-\epsilon)^2} \left(1 + \epsilon + \frac{3}{2}\delta_1 + \frac{\delta_2}{2} + \epsilon^2 + 2\epsilon\delta_1 \right) \\ &= \frac{2}{\eta^2}(1 + 2\epsilon + ...) \left(1 + \epsilon + \frac{3}{2}\delta_1 + ... \right) \\ &\approx \frac{1}{\eta^2}\left(2 + 6\epsilon + 3\delta_1\right). \end{aligned} \tag{3.114}$$

[8]This equation in the Fourier space is

$$\mathcal{R}_k'' + 2\frac{z_k'}{z_k^2}\mathcal{R}_k' + k^2\mathcal{R}_k = 0. \tag{3.107}$$

Consider the second order differential equation:

$$x^2\frac{d^2y}{dx^2} + (2p+1)x\frac{dy}{dx} + \left[\alpha^2 x^{2r} + \beta^2\right] y = 0, \tag{3.115}$$

the solution is

$$y = x^{-p}\left[A\, H^{(1)}_{\frac{q}{r}}\left(\frac{\alpha}{r}x^r\right) + B\, H^{(2)}_{\frac{q}{r}}\left(\frac{\alpha}{r}x^r\right)\right], \quad q = \sqrt{p^2-\beta^2},$$

where A and B are integration constants. $H^{(1)}_{\frac{q}{r}}\left(\frac{\alpha}{r}x^r\right)$ is the Hankel function of the first kind while $H^{(2)}_{\frac{q}{r}}\left(\frac{\alpha}{r}x^r\right) = \left[H^{(1)}_{\frac{q}{r}}\left(\frac{\alpha}{r}x^r\right)\right]^*$ is the Hankel function of the second kind.

In order to find the solution of (3.111), one rewrites it in the form

$$u_k'' + \left[k^2 - \frac{1}{\eta^2}(\nu^2 - \frac{1}{4})\right] u_k = 0$$

$$(k\eta)^2\frac{d^2u_k}{d(k\eta)^2} + \left[(k\eta)^2 - (\nu^2 - \frac{1}{4})\right] u_k = 0. \tag{3.116}$$

where $\nu^2 = \frac{9}{4} + 6\epsilon + 3\delta_1$. It can be seen that the equation (3.116) is in the form of (3.115) with $p = -1/2,\ \alpha = r = 1,\ \beta^2 = -(\nu^2 - 1/4)$ and $q = \nu$. Therefore the solution for (3.116) is given in terms of the Hankel functions:

$$u_k(\eta) = \sqrt{-k\eta}\left[A_k H^{(1)}_\nu(-k\eta) + B_k H^{(2)}_\nu(-k\eta)\right], \tag{3.117}$$

the negative sign shows that η lies in $-\infty < \eta < 0$. The range of the conformal time corresponds to the range of the comoving time as $0 < t < \infty$.

After inflation, the spacetime expands slower than the horizon. Thus perturbations will re-enter the horizon when their wavelengths become smaller than the horizon size. The perturbations re-entered the horizon at the matter-dominated period, since that time they have grown by the gravitational attraction and caused the structure formations and anisotropies in the CMB radiation. These perturbations are universally called *primordial perturbations*.

For convenience, we consider the perturbations in small scale and large scale separately:

Small scale perturbations ($k > aH$)

Small scale perturbations[9] are the perturbations whose wavelengths are smaller than the horizon size at the decoupling time. This means that after crossing

[9] this scale corresponds to both intermediate scale and small scale fluctuations in Chapter II.

outside the horizon during inflation, they have re-entered the horizon before the decoupling time, therefore the perturbations have been changed because of gravity. Considering the equation of motion (3.111) in the small scale limit, $k^2 >> z''/z$, one finds

$$u_k'' + k^2 u_k \approx 0, \tag{3.118}$$

whose solution is a plane wave:

$$u_k(\eta) = D_k e^{-ik\eta}.$$

The constant, D_k, is obtained by using the commutation relation

$$[\varphi(\eta, \vec{x}), \pi(\eta', \vec{x'})]_{\eta=\eta'} = i\delta^{(3)}(\vec{x} - \vec{x'}). \tag{3.119}$$

Here $\varphi \equiv \delta\phi$ is the inflaton field perturbation and $\pi(\eta, \vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = a^2\varphi'$. The relation above offers $|u_k(\eta)| = \frac{1}{\sqrt{2k}}$, where $u = a\varphi + z\psi$ [see appendix C.2]. Thus

$$u_k(\eta) = \frac{1}{\sqrt{2k}}\, e^{-ik\eta}. \tag{3.120}$$

An asymptotic form of the Hankel function is $H_\nu^{(1)}(x \gg 1) \approx \sqrt{\frac{2}{\pi x}} e^{i(x - \nu\frac{\pi}{2} - \frac{\pi}{4})}$. Therefore

$$\lim_{-k\eta \to \infty} H_\nu^{(1)}(-k\eta) \approx \sqrt{\frac{2}{\pi(-k\eta)}}\, e^{-ik\eta} e^{-i\frac{\pi}{2}(\nu + \frac{1}{2})}. \tag{3.121}$$

Substituting it into (3.117) and comparing with (3.120), the integration constants are $B_k = 0$ and $A_k = \frac{1}{2}\sqrt{\frac{\pi}{k}}\, e^{i\frac{\pi}{2}(\nu+\frac{1}{2})}$. The solution which satisfies the short wavelength limit is

$$u_k(\eta) \;=\; \frac{\sqrt{\pi}}{2}\, e^{i(\frac{\pi}{2}(\nu + \frac{1}{2}))} \sqrt{-\eta}\, H_\nu^{(1)}(-k\eta). \tag{3.122}$$

Large scale perturbations ($k < aH$)

Large scale perturbations are the perturbations whose wavelengths are larger than the horizon size at the decoupling time, they have not re-entered the horizon yet therefore their amplitudes are the same as in the inflation period. This can be shown by considering the equation of motion (3.111) in the long wavelength limit, $k^2 << z''/z$, giving

$$u_k'' + \frac{z''}{z} u_k \approx 0,$$

whose solution is

$$u_k(\eta) \sim z_k,$$

or

$$\mathcal{R}_k = \frac{u_k}{z_k} \sim \text{const.},$$

Thus when primordial perturbations on the superhorizon scale re-enter the horizon, the received information is purely the information from inflation.
From $H_\nu^{(1)}(x \ll 1) = -i\frac{\Gamma(\nu)}{\pi}\left(\frac{x}{2}\right)^{-\nu}$, the Hankel function in the long wavelength limit, $-k\eta \to 0$, is

$$H_\nu^{(1)}(-k\eta \to 0) = -i\frac{\Gamma(\nu)}{\pi}\left(\frac{-k\eta}{2}\right)^{-\nu}.$$

$\Gamma(\nu)$ is a gamma function[10]. The equation of motion, (3.122), in this limit is

$$u_k(\eta) \quad \to \quad -i\, e^{i(\frac{\pi}{2}(\nu+\frac{1}{2}))}\, 2^{\nu-\frac{3}{2}}\, \frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\frac{1}{\sqrt{2k}}\,(-k\eta)^{\frac{1}{2}-\nu}. \tag{3.123}$$

Here $\nu = \frac{3}{2}\sqrt{1+\frac{8}{3}\epsilon+\frac{4}{3}\delta_1} \approx \frac{3}{2}(1+\frac{4}{3}\epsilon+\frac{2}{3}\delta_1) = \frac{3}{2}+2\epsilon+\delta_1$. The squared amplitude of the perturbation is

$$|u_k(\eta)|^2 \quad = \quad 2^{2\nu-3}\left[\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right]^2 \frac{(-k\eta)^{1-2\nu}}{2k}. \tag{3.124}$$

Considering the vacuum state, the perturbation amplitude can be expressed as [see appendix C.3]

$$\langle 0|u^*(\eta,\vec{x})u(\eta',\vec{x}')|0\rangle_{\eta=\eta'} \quad = \quad \int \frac{d^3k}{(2\pi)^3}|u_k(\eta)|^2 e^{i\vec{k}(\vec{x}-\vec{x}')}, \tag{3.125}$$

For $\vec{x} = \vec{x}'$, one defines the *power spectrum*, a quantity representing variance of perturbations at a given comoving wavelength, k^{-1}, as

$$\langle 0|u^2(\eta,\vec{x})|0\rangle \quad \equiv \quad \int \frac{dk}{k}\mathcal{P}_u(k). \tag{3.126}$$

Thus the power spectrum of the curvature perturbation is [see appendix C.3]

$$\begin{aligned}\mathcal{P}_{\mathcal{R}}(k) \quad &= \quad \frac{k^3}{2\pi^2}|\mathcal{R}_k(\eta)|^2 \\ &= \quad \frac{1}{2m_{pl}^2\epsilon}\left(\frac{H}{2\pi}\right)^2 2^{2\nu-3}\left[\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right]^2\left(\frac{k}{aH}\right)^{-4\epsilon-2\delta_1}(1-\epsilon)^{2+4\epsilon+2\delta_1}. \end{aligned} \tag{3.127}$$

Because of the consistency conditions, we have small values for the slow roll parameters, $\epsilon \ll 1$ and $\delta_n \ll 1$. One uses the Taylor expansion up to the first order

[10] We use the property: $\Gamma(x+1) = x\Gamma(x)$. From $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2})$ therefore $\sqrt{\pi} = 2\Gamma(\frac{3}{2})$.

in the slow roll parameters to get

$$
\begin{aligned}
2^{\nu-\frac{3}{2}} &= e^{(2\epsilon+\delta_1)\ln 2} \quad \approx 1+(2\epsilon+\delta_1)\ln 2, \\
\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})} &\approx 1+(2-2\ln 2-\gamma)(2\epsilon+\delta_1), \\
(1-\epsilon)^{2+4\epsilon+2\delta_1} &\approx (1-2\epsilon-4\epsilon^2-2\epsilon\delta_1), \\
\left(\frac{k}{aH}\right)^{-4\epsilon-2\delta_1} &= e^{(-4\epsilon-2\delta_1)\ln\left(\frac{k}{aH}\right)} \quad \approx 1-(4\epsilon+2\delta_1)\ln\left(\frac{k}{aH}\right),
\end{aligned} \tag{3.128}
$$

where $\nu = \frac{3}{2}+2\epsilon+\delta_1$ and $\alpha = 2-\ln 2-\gamma = 0.729637$[11]. The power spectrum up to the first order is

$$
\mathcal{P}_{\mathcal{R}}(k) = \frac{1}{2m_{pl}^2\epsilon}\left(\frac{H}{2\pi}\right)^2\left[1-2\epsilon+2\left(\alpha-\ln\left(\frac{k}{aH}\right)\right)(2\epsilon+\delta_1)\right]. \tag{3.129}
$$

In addition, we can measure the amplitudes of the perturbations after the perturbations re-enter the horizon. In order to receive only the signal from inflation, without other sources, it is necessary to measure the perturbations amplitudes when they re-enter the horizon immediately. After that their amplitudes will change due to gravitational instabilities. Therefore we calculate the *primordial* power spectrum when the *primordial* perturbation wavelength is equal to the horizon size: $k = aH$.

$$
\begin{aligned}
\mathcal{P}_{\mathcal{R}}(k) &= \lim_{k\to aH}\frac{k^3}{2\pi^2}\left|\frac{u_k(\eta)}{z}\right|^2 \\
&= \frac{1}{2m_{pl}^2\epsilon}\left(\frac{H}{2\pi}\right)^2\left[1-2\epsilon+2\alpha(2\epsilon+\delta_1)\right].
\end{aligned} \tag{3.130}
$$

The spectral index is defined by the first derivative of the power spectrum with respect to the scale k

$$
n_{\mathcal{R}}(k) = 1+\frac{d\ln\mathcal{P}_{\mathcal{R}}}{d\ln k}. \tag{3.131}
$$

The scale invariant spectrum occurs when $n_{\mathcal{R}}(k) = 1$ which means that the power spectrum does not depend on the scale, k.

Before computing the spectral index, one calculates $d\ln k$ up to the first order in the slow roll parameters at the horizon crossing scale, and obtains

$$
d\ln k = (1-\epsilon)Hdt \;=\; (1-\epsilon)d\ln a. \tag{3.132}
$$

The derivatives of the slow roll parameters with respect to the comoving time are

$$
\dot{\epsilon} = 2H(\epsilon^2+\epsilon\delta_1),\;\; \dot{\delta_1} = H(\epsilon\delta_1-\delta_1^2+\delta_2),\;\; \dot{\delta_2} = H(2\epsilon\delta_2-\delta_1\delta_2+\delta_3). \tag{3.133}
$$

[11] $\gamma = \lim_{n\to\infty}(1+\frac{1}{2}+...+\frac{1}{n}-\ln n) = 0.577216$ is the Euler-Mascheroni constant.

Using (3.133) and (3.132), one obtains

$$\frac{d\epsilon}{d\ln k} \simeq 2(\epsilon^2 + \epsilon\delta_1), \ \frac{d\delta_1}{d\ln k} \simeq (\epsilon\delta_1 - \delta_1^2 + \delta_2), \ \frac{d\delta_2}{d\ln k} \simeq (2\epsilon\delta_2 - \delta_1\delta_2 + \delta_3). \quad (3.134)$$

The spectral index up to the second order in the slow roll parameter is [see appendix C.4]

$$n_{\mathcal{R}}(k) = 1 - 4\epsilon - 2\delta_1 + (8\alpha - 4)\epsilon^2 + (10\alpha - 4)\epsilon\delta_1 - 2\alpha\delta_1^2 + 2\alpha\delta_2. \quad (3.135)$$

The running of the spectral index is $\frac{dn_{\mathcal{R}}}{d\ln k}$. Its value up to the third order is

$$\begin{aligned} \frac{dn_{\mathcal{R}}}{d\ln k}(k) \ &= \ -8\epsilon^2 - 10\epsilon\delta_1 + 2\delta_1^2 - 2\delta_2 + (32\alpha - 16)\epsilon^3 + (62\alpha - 28)\epsilon^2\delta_1 \\ &\quad +(6\alpha - 4)\epsilon\delta_1^2 + (14\alpha - 4)\epsilon\delta_2 + 4\alpha\delta_1^3 - 6\alpha\delta_1\delta_2 + 2\alpha\delta_3. \quad (3.136) \end{aligned}$$

The last two equations and the power spectrum depend on the inflation models because the slow roll parameters depend on the inflaton potential. The spectral index and its running in several models will be computed in the following chapter. After that their values will be compared with the WMAP data.

CHAPTER IV

SINGLE-FIELD INFLATION

This chapter concerns the power-law inflation driven by a single inflaton field in both commutative and noncommutative spacetime. We will show problems of the simplest model of the commutative inflation and see how they can be solved by the existence of a minimum length scale.

4.1 Commutative Inflation

In this model, the universe is in the commutative spacetime therefore the only one effect coming from the spacetime is gravity (curvature).

The power-law inflation has the potential as below

$$V(\phi) = V_0 \exp\Big(-\sqrt{\frac{2}{p}}\phi\Big), \tag{4.1}$$

where $p > 1$ gives the condition of accelerated expansion. The scale factor in this model has the form

$$a(t) \sim t^p. \tag{4.2}$$

One calculates the Hubble parameter and the slow roll parameters, and obtains

$$H = \frac{p}{t},\ \epsilon = \frac{1}{p},\ \delta_1 = -\frac{1}{p},\ \delta_2 = \frac{2}{p^2},\ \delta_3 = -\frac{6}{p^3}. \tag{4.3}$$

The spectral index and the running for this model are

$$n_{\mathcal{R}}(k) = 1 - \frac{2}{p}, \tag{4.4}$$

$$\frac{dn_{\mathcal{R}}}{d\ln k} = 0. \tag{4.5}$$

Comparing the obtained values with the WMAP data

$$n_{\mathcal{R}} = 0.93 \pm 0.03,\quad \frac{dn_{\mathcal{R}}}{d\ln k} = -0.031^{+0.016}_{-0.017}\ \ (k = 0.05\mathrm{Mpc}^{-1}),$$

$$n_{\mathcal{R}} = 1.20^{+0.12}_{-0.11},\quad \frac{dn_{\mathcal{R}}}{d\ln k} = -0.077^{+0.050}_{-0.052}\ \ (k = 0.002\mathrm{Mpc}^{-1}),$$

the power-law inflation gives only the scale invariant spectral index with red-tilt, $n_{\mathcal{R}} < 1$, and the zero running for all exponent, p. The results are not consistent with the WMAP data.

4.2 Noncommutative Inflation

The idea of the noncommutative spacetime comes from the string theory. In string theory, there is the existence of the minimum length scale called string length, l_s (string is an one dimensional object). If we use it as a ruler, anything whose length scale is smaller than the string scale cannot be measured correctly.

At high energies in the early universe and during the inflation period, quantum effects are expected to be important. From the string theory, the primordial perturbations whose wavelengths are closed to the string scale are affected by the spacetime noncommutativity.

To probe these effects, the WMAP spectrum of CMB anisotropies is used. We have already known that the anisotropies of the cosmic microwave background radiation come from the primordial curvature perturbations generated during inflation period. When we observe the CMB anisotropies, we hope to get some information on spacetime noncommutativity using noncommutative inflation models.

4.2.1 Noncommutative Modifications to the Perturbation Equations of Motion

The universal property of the string theory is the stringy spacetime uncertainty relation proposed by Brandenberger and Ho [13]:

$$\Delta t_p \Delta x_p \geq l_s^2, \tag{4.6}$$

where t_p and x_p are physical time and space respectively. This relation implies that the space and time at the very short distance near the string scale are noncommutative.

Considering the lowest limit, one finds the relation in the comoving coordinate[1] $(t, \vec{x}(t))$ where $\Delta t_p = \Delta t$ and $\Delta \vec{x}_p = a(t)\Delta \vec{x}$.

$$[t, x] = i\frac{2l_s^2}{a}. \tag{4.7}$$

[1]because we live in the expanding universe.

This commutation relation is time-dependent because the scale factor in the right hand side is a function of time. When the time changes, both the scale factor and the time interval change. The right hand side of the above equation is not constant. From this reason, Brandenberger and Ho introduce the new time coordinate, τ and apply the stringy spacetime uncertainty relation in cosmology:

$$\big[\tau, x\big]_* = i2l_s^2, \tag{4.8}$$

where the $*$-commutator in the above equation is defined by $\big[\tau, x\big]_* \equiv \tau * x - x * \tau$. The $*$-product of any $f(x,\tau)$ and $g(x,\tau)$ functions can be defined as [13]

$$(f * g)(x,\tau) = e^{-il_s^2(\partial_\tau\partial_x - \partial_x\partial_\tau)} \cdot f(x,\tau)g(y,\tau')\big|_{y=x,\tau=\tau'}. \tag{4.9}$$

The $*$-operator maps all multiplications in the noncommutative spacetime into the $*$-product in the commutative spacetime. τ and x are coordinates in the FRW metric:

$$ds^2 \;\; = a^{-2}(\tau)d\tau^2 - a^2(\tau)dx^2 \;\; = dt^2 - a^2(t)dx^2, \tag{4.10}$$

so $d\tau = adt$.

There is the difficulty for considering the noncommutative effect in the cosmological background because noncommutativity will break the homogeneity and isotropy of the universe. So, one considers the noncommutative effect in the momentum space. First, one calculates the modified action of a free scalar field, ϕ, in 1+1 dimensional noncommutative spacetime, then extends it into 3+1 spacetime.

The modified action in 1+1 noncommutative spacetime is (One places $(\tilde{\ })$ over all parameters in the noncommutative spacetime.)

$$\tilde{S} \;\; = \;\; \int d\tau dx \frac{1}{2}\Big(\partial_\tau\tilde{\phi}^\dagger * a^2 * \partial_\tau\tilde{\phi} - \big(\partial_x\tilde{\phi}\big)^\dagger * a^{-2} * \partial_x\tilde{\phi}\Big). \tag{4.11}$$

The Fourier transform of $\tilde{\phi}(\tau,x)$ is $\tilde{\phi}(\tau,k) \equiv \tilde{\phi}_k$ where

$$\tilde{\phi} = V^{\frac{1}{2}}\Big[\frac{1}{2}\int\frac{dk}{\sqrt{2\pi}}\Big(\tilde{\phi}_k e^{ikx} + \tilde{\phi}_k^\dagger e^{-ikx}\Big)\Big], \tag{4.12a}$$

$$\tilde{\phi}^\dagger = V^{\frac{1}{2}}\Big[\frac{1}{2}\int\frac{dq}{\sqrt{2\pi}}\Big(\tilde{\phi}_q^\dagger e^{-iqx} + \tilde{\phi}_q e^{iqx}\Big)\Big], \tag{4.12b}$$

with the total spatial coordinate volume, V. The condition for the real value of $\tilde{\phi}$ is $\tilde{\phi}_k^\dagger = \tilde{\phi}_{-k}$. One substitutes (4.12) into (4.11), then uses the $*$-product, the action for the time-time component is [see appendix C.5]

$$\tilde{S}_{time} \;\; = \;\; V\int_{|k|<k_0} d\tau dk \frac{1}{2}\partial_\tau\tilde{\phi}_{-k}\,\partial_\tau\tilde{\phi}_k\,\left[\frac{a^2(\tau + kl_s^2) + a^2(\tau - kl_s^2)}{2}\right].$$

From the effect of the spacetime noncommutativity, the scale factor is a function of time and the scale k [2]. The cut-off momentum k_0 comes from the stringy uncertainty relation in the lowest limit $\Delta\tau\Delta x = l_s^2$. The lower bound of length corresponds to the upper bound of the momentum $k_0 = a_{eff}/l_s$. Here, a_{eff} is the effective scale factor with $a_{eff}(t) \to a(t)$ when $l_s \to 0$.

In the same way, the action for the space-space component is

$$\tilde{S}_{space} \;=\; -V\int_{|k|<k_0} d\tau dk \frac{1}{2}k^2\tilde{\phi}_{-k}\,\tilde{\phi}_k\,\left[\frac{a^{-2}(\tau+kl_s^2)+a^{-2}(\tau-kl_s^2)}{2}\right].$$

The full action is

$$\tilde{S} \;=\; V\int_{|k|<k_0} d\tau dk \frac{1}{2}\Big(\beta_k^+\partial_\tau\tilde{\phi}_{-k}\partial_\tau\tilde{\phi}_k - k^2\beta_k^-\tilde{\phi}_{-k}\tilde{\phi}_k\Big), \tag{4.13}$$

where

$$\beta_k^{\pm} = \frac{1}{2}\Big(a^{\pm 2}(\tau - l_s^2 k) + (a^{\pm 2}(\tau + l_s^2 k)\Big). \tag{4.14}$$

The action above is rewritten in a conformal time coordinate by defining [13]

$$d\tilde{\eta} = a_{eff}^{-2}\; d\tau, \tag{4.15}$$

where

$$a_{eff}^2 = \left(\frac{\beta_k^+}{\beta_k^-}\right)^{1/2}. \tag{4.16}$$

Here, $\tilde{\eta}$ is called the *modified* conformal time coordinate. Defining $y_k = (\beta_k^+\beta_k^-)^{1/4}$, the full action in this coordinate is

$$\tilde{S} \;=\; V\int_{|k|<k_0} d\tilde{\eta}\; dk\; \frac{1}{2}y_k^2(\tilde{\eta})\Big(\tilde{\phi}'_{-k}\tilde{\phi}'_k - k^2\tilde{\phi}_{-k}\tilde{\phi}_k\Big). \tag{4.17}$$

Next, one generalizes the action to $d+1$ spacetime,

$$\tilde{S} \;=\; V\int_{|k|<k_0} d\tilde{\eta}\; d^dk\; \frac{1}{2}z_k^{d-1}(\tilde{\eta})\Big(\tilde{\phi}'_{-k}\tilde{\phi}'_k - k^2\tilde{\phi}_{-k}\tilde{\phi}_k\Big),$$

with $z_k^{d-1}(\tilde{\eta},k) = z^{d-1}(\tilde{\eta})y_k^2(\tilde{\eta})$. In $3+1$ spacetime, where we live in, the action is

$$\tilde{S} \;=\; V\int_{|k|<k_0} d\tilde{\eta}\; d^3k\; \frac{1}{2}z_k^2(\tilde{\eta})\Big(\tilde{\phi}'_{-k}\tilde{\phi}'_k - k^2\tilde{\phi}_{-k}\tilde{\phi}_k\Big). \tag{4.18}$$

From the action principle, the equation of motion for $\tilde{\phi}$ is

$$\tilde{\phi}''_k \,+\, 2\frac{z'_k}{z_k^2}\tilde{\phi}'_k \,+\, k^2\tilde{\phi}_k = 0. \tag{4.19}$$

[2]In commutative models, $a(t)$ depends only on time.

Here, ϕ is any free scalar field which is affected by the spacetime noncommutativity in the high energies era. The inflaton field in this model is also affected by the uncertainty because the inflation era occurs in the early universe. According to the previous chapter, the equation above is the same as the equation of motion of the curvature perturbation, containing the inflaton fluctuation and the metric fluctuation. It can be seen that in the case of inflation, $\tilde{\phi} = \tilde{\mathcal{R}}$ [14].

4.2.2 Noncommutative Parameter and the Noncommutative Power Spectrum

Again, one eliminates the friction term by defining

$$\tilde{u}_k(\tilde{\eta}) = z_k \tilde{\mathcal{R}}_k(\tilde{\eta}) \tag{4.20}$$

The equation of motion for $\tilde{u}_k$ is

$$\tilde{u}_k'' + \left(k^2 - \frac{z_k''}{z_k}\right)\tilde{u}_k = 0. \tag{4.21}$$

The noncommutative models have the same form of the equation of motion as commutative models except that z_k depends on the modified conformal time and scale k [(3)]. The difference leads to the different value of the power spectrum.

One defines the noncommutative parameter, μ, as [16]

$$\mu = \left(\frac{kH}{aM_s^2}\right)^2 \tag{4.22}$$

which measures the ratio of the Hubble radius at the horizon exit and the string length, $l_s \equiv M_s^{-1}$. From $y_k = (\beta_k^+ \beta_k^-)^{1/4}$, the relation between the noncommutative parameter and y_k is

$$y_k \approx 1 + \mu, \tag{4.23}$$

this relation will be used to find the power spectrum and show its suppression on the large scale.

From (4.16) and $H = da/d\tau$, one obtains

$$\begin{aligned} a_{eff}^2 &= \left(\frac{\beta_k^+}{\beta_k^-}\right)^{1/2} \\ &= a(\tau + kl_s^2)\cdot a(\tau - kl_s^2) \\ &= \left[a + \frac{da}{d\tau}(kl_s^2) + \frac{1}{2}\frac{d^2a}{d\tau^2}(kl_s^2)^2 + ...\right]\left[a - \frac{da}{d\tau}(kl_s^2) + \frac{1}{2}\frac{d^2a}{d\tau^2}(kl_s^2)^2 - ...\right] \\ &\approx a^2(1-\mu). \end{aligned} \tag{4.24}$$

[3] In commutative models, $z(\eta)$ depends only on the conformal time.

$d\tilde{\eta}/d\tau$ is written in terms of μ as

$$d\tau = a^2(1-\mu)d\tilde{\eta} \tag{4.25}$$

whereas $d\tau = a^2 d\eta$ in the commutative case. The relation between the conformal time and the modified conformal time is

$$d\eta = (1-\mu)d\tilde{\eta}. \tag{4.26}$$

Then as in Chapter III, z_k''/z_k can be written in terms of the slow roll parameters and the noncommutative parameter as

$$\begin{aligned}
\frac{1}{z_k}\frac{d^2 z_k}{d\tilde{\eta}} &= \frac{1}{z_k}\frac{d^2 z_k}{d\eta^2}(1-\mu^2) \\
&\approx \frac{1}{z}\frac{d^2 z}{d\eta^2}(1-2\mu) \\
&= 2(aH)^2\left(1+\epsilon+\frac{3}{2}\delta_1-2\mu\right),
\end{aligned} \tag{4.27}$$

where $z''/z = 2(aH)^2\left(1+\epsilon+\frac{3}{2}\delta_1\right)$ in the commutative case.
One determines the horizon crossing scale from the condition $k^2 = z_k''/z_k$ [13], which is the pivot scale in the equation of motion (4.21).

$$k^2 = \frac{z_k''}{z_k} = 2(aH)^2\left(1+\epsilon+\frac{3}{2}\delta_1-2\mu\right) \tag{4.28}$$

$$k \approx \sqrt{2}aH. \tag{4.29}$$

From (4.26) and $\eta = -[aH(1-\epsilon)]^{-1}$

$$aH \approx \frac{-1}{\tilde{\eta}}(1+\epsilon+\mu). \tag{4.30}$$

Considering the time when the fluctuation mode k crosses outside the Hubble radius, one finds $\tilde{\eta} = \left(\frac{1+\epsilon+\mu}{1+\epsilon}\right)\eta > \eta$. This means that the spacetime uncertainty delays the exit time of fluctuations. Comparing with the commutative case, the perturbation amplitudes at the horizon crossing are changed due to the delay of the exit time.

The solution $u_k(\tilde{\eta})$ of (4.21) looks very much like the equation (3.123) in Chapter III because of the same form of the equation of motion. The power spectrum determined at the new horizon crossing scale, $k = \sqrt{2}aH$, is

$$\begin{aligned}
\tilde{\mathcal{P}}_{\mathcal{R}}(k) &= \lim_{k\to\sqrt{2}aH}\frac{k^3}{2\pi^2}\left|\frac{u_k(\tilde{\eta})}{z_k}\right|^2 \\
&= \lim_{k\to\sqrt{2}aH}\frac{1}{z_k^2}2^{2\nu-3}\left[\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right]^2\frac{(-k\tilde{\eta})^{1-2\nu}}{2k}.
\end{aligned}$$

Substituting the equations (4.23), (4.30), $z_k^2 = z^2 y_k^2$ and $\nu = \frac{3}{2} + 2\epsilon + \delta_1$ in the above equation, then the power spectrum yields

$$\begin{aligned}\tilde{\mathcal{P}}_{\mathcal{R}}(k) &= \lim_{k\to\sqrt{2}aH} \frac{1}{2m_{pl}^2\epsilon}\left(\frac{H}{2\pi}\right)^2 2^{2\nu-3}\left[\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right]^2\left[\frac{k}{aH}\right]^{-4\epsilon-2\delta_1} \\ &\quad \cdot\frac{1}{(1+\epsilon+\mu)^{2+4\epsilon+2\delta_1}(1+\mu)^2} \\ &\approx \mathcal{P}_{\mathcal{R}}(k)\frac{1}{(1+\mu)^{4+4\epsilon+2\delta_1}}. \end{aligned} \tag{4.31}$$

It can be seen that the power spectrum of the primordial perturbations is suppressed by the noncommutative effect by the factor $\frac{1}{(1+\mu)^4}$ approximately. However, the suppression occurs only at the large scale when μ is large enough to be important (will be shown later).

Using the Taylor expansion in appendix C.3, one calculates the power spectrum up to the first order in the slow roll parameters

$$\begin{aligned}\tilde{\mathcal{P}}_{\mathcal{R}}(k) &= \lim_{k\to\sqrt{2}aH}\left\{\frac{1}{2m_{pl}^2\epsilon}\left(\frac{H}{2\pi}\right)^2\left[1-2\epsilon+2\left(\alpha-\ln\left(\frac{k}{aH}\right)\right)(2\epsilon+\delta_1)\right]\right. \\ &\quad -\frac{\mu}{2m_{pl}^2\epsilon}\left(\frac{H}{2\pi}\right)^2\left[4+16\epsilon\left(\alpha-\ln\left(\frac{k}{aH}\right)\right)+8\epsilon\right. \\ &\quad \left.\left.+8\delta_1\left(\alpha-\ln\left(\frac{k}{aH}\right)\right)+2\delta_1\right]\right\} \\ &\approx \mathcal{P}_{\mathcal{R}}(k)-\frac{\mu}{2m_{pl}^2\epsilon}\left(\frac{H}{2\pi}\right)^2\left[4+(16\alpha_*+8)\epsilon+(8\alpha_*+2)\delta_1\right], \end{aligned} \tag{4.32}$$

where the commutative contribution (the first term of the right hand side) is now evaluated at $k = \sqrt{2}aH$ which is the same as changing α to be $\alpha_* = \alpha - \frac{\ln 2}{2} = 0.3831$. All additional terms depend on the noncommutative parameter, so the power spectrum reduces to the commutative power spectrum when $\mu \to 0$ (no minimum length scale).

Using the definition of μ, one finds its derivatives at $k = \sqrt{2}aH$ [see appendix C.4 and (3.132)]

$$\dot{\mu} = -4H\mu\epsilon, \quad \frac{d\mu}{d\ln k} \approx -4\mu\epsilon. \tag{4.33}$$

The spectral index up to the second order in the slow roll parameters is

$$\begin{aligned}\tilde{n}_{\mathcal{R}}(k) &= n_{\mathcal{R}}(k) + 16\mu\epsilon \\ &\quad + \mu\left[(32\alpha_*+16)\epsilon^2 - (8\alpha_*+10)\epsilon\delta_1 + (8\alpha_*+2)(\delta_1^2-\delta_2)\right]. \end{aligned} \tag{4.34}$$

The running of the spectral index up to the third order is

$$\begin{aligned}\frac{d\tilde{n}_{\mathcal{R}}(k)}{d\ln k} &= \frac{dn_{\mathcal{R}}(k)}{d\ln k} - 32\mu\epsilon(\epsilon-\delta_1) \\ &- \mu\left[-(136\alpha_* + 74)\epsilon^2\delta_1 + (24\alpha_* + 14)\epsilon\delta_1^2 - (8\alpha_* - 6)\epsilon\delta_2 \right. \\ &\left. +(8\alpha_* + 2)(2\delta_1^3 - 3\delta_1\delta_2 + \delta_3)\right].\end{aligned} \tag{4.35}$$

The power spectrum, the spectral index and the running are reduced to be those obtained in the commutative case when $l_s \to 0$.

4.2.3 Noncommutative Power-Law Inflation

The power-law inflation gives

$$a(t) = a_i t^p, \tag{4.36}$$

where a_i is the value of the scale factor at the beginning of inflation. The slow roll parameters in this model are

$$\epsilon = \frac{1}{p},\ \delta_1 = -\frac{1}{p},\ \delta_2 = \frac{2}{p^2},\ \delta_3 = -\frac{6}{p^3}. \tag{4.37}$$

One rewrites μ in terms of the exponent p by integrating the equation (4.33)

$$\mu(k) = \left(\frac{k}{k_c}\right)^{-4\epsilon} = \left(\frac{k_c}{k}\right)^{\frac{4}{p}}, \tag{4.38}$$

where k_c is the integration constant which is the lowest limit of k. The small scale limit where $k \gg k_c$ leads to $\mu \to 0$ giving the same power spectrum as the commutative inflation.

In order to determine the inflation parameter p and the scale k_c, we use the data at $k = 0.05\text{Mpc}^{-1}$. Later, we make a prediction about the spectral index and its running at $k = 0.002\text{Mpc}^{-1}$.

By comparison to the recent data at $k = 0.05\text{Mpc}^{-1}$, the best-fit values of the model parameters are

$$p = 12.171, \quad k_c = 9.82 \times 10^{-6}\text{Mpc}^{-1}. \tag{4.39}$$

Next, the spectral index and its running at $k = 0.002\text{Mpc}^{-1}$ are predicted to be

$$\tilde{n}_{\mathcal{R}} = 1.11, \qquad \frac{d\tilde{n}_{\mathcal{R}}}{d\ln k} = -0.089 \tag{4.40}$$

whereas the commutative power-law inflation with the same exponent and $l_s = 0$ gives

$$n_{\mathcal{R}} = 0.836, \qquad \frac{dn_{\mathcal{R}}}{d\ln k} = 0. \tag{4.41}$$

It can be seen that the predicted values from the noncommutative inflation are quite closed to those of the WMAP, especially the spectral index. In addition, the noncommutative power-law inflation can give the blue-tilted spectrum for the large scale, and allows for the running of the spectral index.

For determining the string scale, l_s, the significant problem is that μ is implicitly time-dependent. It is necessary to know the exact time when fluctuations of mode k cross outside the horizon. Some papers, such as [17], use the ending time of inflation to be the exit time. This method provides the correct value of l_s only for the perturbation mode that crosses outside the horizon at the end of inflation. Another choice is using the value of the power spectrum. This approach gives the same order of magnitude $l_s \sim 10^{-29}$ cm. for the string scale from whatever scale of k we use.

For elucidating the results above, some new parameters are needed.

From (4.36), and $d\tau = adt$. One finds

$$a(\tau) = a_i^{\frac{1}{p+1}}[(p+1)\tau]^{\frac{p}{p+1}}. \tag{4.42}$$

$a(\tau)$ is dimensionless, one defines an inflation scale, l, as

$$\begin{aligned} l^{\frac{p}{p+1}} &= \left[a_i^{\frac{1}{p+1}}(p+1)^{\frac{p}{p+1}}\right]^{-1} \\ l &= \frac{1}{a_i^{1/p}(p+1)}. \end{aligned} \tag{4.43}$$

The scale factor and the Hubble parameter in terms of the inflation scale are

$$a(\tau) = \left(\frac{\tau}{l}\right)^{\frac{p}{p+1}}, \quad a(t) = \left[\frac{t}{l(p+1)}\right]^p, \tag{4.44}$$

$$H(\tau) = \left(\frac{p}{p+1}\right)(l^p\tau)^{\frac{-1}{p+1}}. \tag{4.45}$$

Using the cut-off momentum $k = k_0 = a_{eff}/l_s$, the time τ can be written in terms of k as

$$\tau = kl_s^2\left[1+\left(\frac{k}{k_s}\right)^{\frac{p}{2}}\right]^{\frac{1}{2}} \tag{4.46}$$

with $k_s = l_s^{p-1}/l^p$ i.e. the ratio of the string scale and the inflation scale. When $k \gg k_s$[4], one finds

$$\tau \approx l(kl_s)^{\frac{p}{p+1}}. \tag{4.47}$$

[4]The condition is in the UV region. In the UV region, the string energy scale is much larger than the cosmological energy scale (here is the inflation scale) when perturbations are generated. This region gives $l \gg l_s$.

Table 4.1: Comparison of important quantities between commutative and noncommutative inflation models.

	Commutative Inflation	Noncommutative Inflation
Action	$S = V\int d\eta\, d^3k\, \frac{1}{2}z^2(\eta) \left(\mathcal{R}'_{-k}\mathcal{R}'_k - k^2\mathcal{R}_{-k}\mathcal{R}_k\right)$	$\tilde{S} = V\int_{\lvert k\rvert<k_0} d\tilde{\eta}\, d^3k\, \frac{1}{2}z_k^2(\tilde{\eta}) \left(\tilde{\mathcal{R}}'_{-k}\tilde{\mathcal{R}}'_k - k^2\tilde{\mathcal{R}}_{-k}\tilde{\mathcal{R}}_k\right)$
Equation of motion	$u''_k + \left(k^2 - \frac{z''}{z}\right)u_k = 0$	$\tilde{u}''_k + \left(k^2 - \frac{z''_k}{z_k}\right)\tilde{u}_k = 0$
Power spectrum	$\mathcal{P}_{\mathcal{R}}(k) = \frac{1}{2m_{pl}^2\epsilon}\left(\frac{H}{2\pi}\right)^2 [1 - 2\epsilon + 2\alpha(2\epsilon + \delta_1)]$	$\tilde{\mathcal{P}}_{\mathcal{R}}(k) = \mathcal{P}_{\mathcal{R}}(k) - \frac{\mu}{2m_{pl}^2\epsilon}\left(\frac{H}{2\pi}\right)^2 [4 + (16\alpha_* + 8)\epsilon + (8\alpha_* + 2)\delta_1]$
Horizon crossing scale	$k = aH$	$k = \sqrt{2}aH$
Spectral index	$n_{\mathcal{R}}(k) = 1 - 4\epsilon - 2\delta_1 + (8\alpha - 4)\epsilon^2 + (10\alpha - 4)\epsilon\delta_1 - 2\alpha\delta_1^2 + 2\alpha\delta_2$	$\tilde{n}_{\mathcal{R}}(k) = n_{\mathcal{R}}(k) + 16\mu\epsilon + \mu[(32\alpha_* + 16)\epsilon^2 - (8\alpha_* + 10)\epsilon\delta_1 + (8\alpha_* + 2)(\delta_1^2 - \delta_2)]$
Running of spectral index	$\frac{dn_{\mathcal{R}}}{d\ln k}(k) = -8\epsilon^2 - 10\epsilon\delta_1 + 2\delta_1^2 - 2\delta_2 + (32\alpha - 16)\epsilon^3 + (62\alpha - 28)\epsilon^2\delta_1 + 4\alpha\delta_1^3 + (6\alpha - 4)\epsilon\delta_1^2 - 6\alpha\delta_1\delta_2 + (14\alpha - 4)\epsilon\delta_2 + 2\alpha\delta_3$	$\frac{d\tilde{n}_{\mathcal{R}}(k)}{d\ln k} = \frac{dn_{\mathcal{R}}(k)}{d\ln k} - 32\mu\epsilon(\epsilon - \delta_1) - \mu[-(136\alpha_* + 74)\epsilon^2\delta_1 + (24\alpha_* + 14)\epsilon\delta_1^2 - (8\alpha_* - 6)\epsilon\delta_2 + (8\alpha_* + 2)(2\delta_1^3 - 3\delta_1\delta_2 + \delta_3)]$

From (4.45) and (4.47), one rewrites the scale k as

$$k = \frac{1}{l_s}\left[\frac{p}{Hl(p+1)}\right]^p. \quad (4.48)$$

Comparing the equation (4.38) with (4.22) at the horizon crossing, $k = \sqrt{2}aH$, then we have the relation between k_c and k_s in the limit of large p for which $p \pm 1 \approx p$ [18]

$$k_c = \left[\frac{p(2p-1)}{(p+1)^2}\right]^{\frac{p+1}{4}} k_s. \quad (4.49)$$

Substituting $\mathcal{P}_{\mathcal{R}}(k)$ from Chapter III (with $\alpha \to \alpha_*$) into (4.32) and using the value of the power spectrum at $k = 0.05\text{Mpc}^{-1}$, we can solve for the inflation energy scale[5] related to the Planck energy scale:

$$H = 1.54 \times 10^{-4} m_{pl}. \quad (4.50)$$

Remembering that $m_{pl} \equiv \frac{1}{\sqrt{8\pi G}}$ is the reduced Planck mass[6]. The noncommutative parameter at the horizon exit is

$$\mu = \left(\frac{kH}{aM_s^2}\right)^2 = 2\left(\frac{H}{M_s}\right)^4.$$

Computing μ at the same scale by using (4.38), the string mass and the string length are found to be

$$\begin{aligned} M_s &= 3.68 \times 10^{-4} m_{pl}, \\ l_s &= 2.19 \times 10^{-29}\text{cm}. \end{aligned} \quad (4.51)$$

Similarly, the string scale obtained from the power spectrum at the cluster scale is $l_s = 1.87 \times 10^{-29}$ cm, which is the same order as that obtained from the galactic scale.

Furthermore, the inflation scale can be computed by (4.49) and the definition of k_s:

$$\begin{aligned} k_s &= 1.94 \times 10^{-6}\text{Mpc}^{-1} = 6.64 \times 10^{-31}\text{cm}, \\ l &= 1.49 \times 10^{-24}\text{cm}. \end{aligned} \quad (4.52)$$

Note that 1 Mpc $= 3.086 \times 10^{24}$ cm.

[5]The Hubble time, $H^{-1} = a/\dot{a}$, represents the timescale of the evolution of $a(t)$, and the energy scale is the inverse timescale. Therefore, H can be considered as the inflation energy scale [22].

[6]$m_{pl} = 2.436 \times 10^{18}\text{GeV} = 1.235 \times 10^{32}\text{cm}^{-1}$.

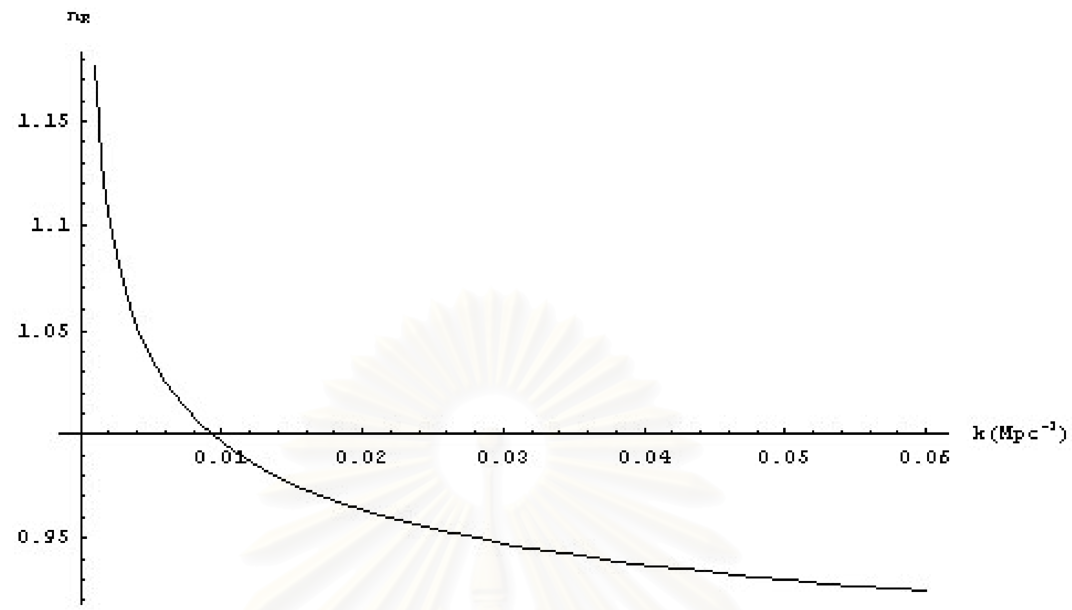

Figure 4.1: $n_{\mathcal{R}}$ as a function of k with $p = 12.171$ and $k_c = 9.82 \times 10^{-6}\text{Mpc}^{-1}$.

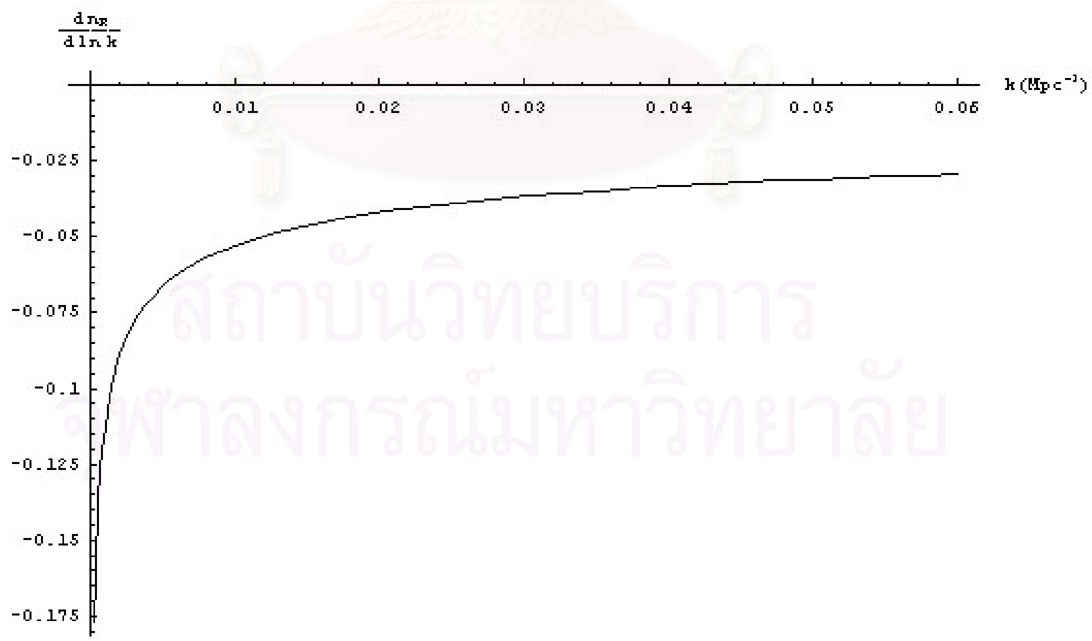

Figure 4.2: $\frac{dn_{\mathcal{R}}}{d\ln k}$ as a function of k with $p = 12.171$ and $k_c = 9.82 \times 10^{-6}\text{Mpc}^{-1}$.

We summarize that the spacetime noncommutativity suppresses the power spectrum on the large scale. The existence of the string length causes the delay of time when perturbations cross outside the horizon, so their amplitudes are changed, as well as the power spectrum, spectral index and the running as shown in Table 4.1. Figure 4.1 shows that the spacetime uncertainty gives the blue-tilted spectrum for the cluster scale and the red one for the galactic scale with the pivot scale $k = 0.0093$ Mpc^{-1}. Furthermore, noncommutative inflation allows for the negative running of the spectral index as shown in Figure 4.2. All results are consistent with the recent WMAP data.

CHAPTER V

MULTI-FIELD INFLATION

The chapter concerns inflation model driven by multiple scalar fields in the commutative spacetime only. From Chapter III, the single-field inflation gives the constant amplitude of the comoving curvature perturbation, $\mathcal{R}$, when it leaves the horizon. The perturbation is purely adiabatic. However, there remain problems as follow:

- No origin of the cold dark matter (CDM)[1].
- Slow roll approximations give the unnatural flat potential.
- Single-field models cannot give some kind of perturbations. According to the observations 'Temperature-Polarization angular power spectrum' [5], the single-field models can generate only the adiabatic perturbations.
- Single-field models usually give a zero running of the spectral index which is not consistent with observations.

Some problem, such as the last one, can be solved by considering inflation on the noncommutative spacetime as we have already discussed in the previous chapter. However, adding other fields is another way to solve the rest.

Multi-field inflation models can be separated into two classes: 1. *Multiple inflation*, where there are more than one inflaton fields driving inflation giving rise to multiple inflationary stages. 2. *N-field inflation*[2] where only one of the N

[1] Dark matter is the extra material whose interaction is only gravity and emits no detectable radiation. The total mass of the visible matter is less than 10 percent of the mass of the dark matter. Cold dark matter, one of the dark matter candidates, is the non-relativistic dark matter which clumps into small regions.

[2] The second case corresponds to the particle physics point of view telling us that there are many kinds of particles in the universe. However, this model is quite similar to the single field inflation except that there are some auxiliary fields ending inflation. Thus, this chapter concerns only the first case and its different results.

scalar fields acts as the inflaton. The other $N-1$ scalar fields have less energy densities than that of the inflaton in order to have the inflation-dominated period.

The existence of more than one fields and their own fluctuations causes the relative fluctuations among themselves. This produces another kind of primordial perturbations.

5.1 Classification of Primordial Perturbations

Primordial perturbations can be classified into two kinds:

- Curvature/adiabatic perturbations ($\mathcal{R}$)
  Adiabatic perturbation is the perturbation in the total energy density of the universe. The spatial distribution of each species is the same:

$$\frac{\delta\rho}{\rho} = \frac{\delta\rho_x}{\rho_x} = \frac{\delta\rho_y}{\rho_y}. \tag{5.1}$$

  Here ρ is the total energy density of the universe, ρ_x and ρ_y are energy densities of any species x and y in the universe. In the field space, the adiabatic perturbation perturbs the trajectory back and forth along the background trajectory.

  From the Einstein equation, this perturbation also perturbs the curvature, as well as the expansion rate of the universe. Thus it is called the *curvature* perturbation.
  The curvature perturbation for N scalar fields is defined by [26]

$$\mathcal{R} \equiv \psi + \frac{H}{\rho+p}(\sum_{i=1}^{N}\dot{\phi}_i\delta\phi_i). \tag{5.2}$$

  In the case of N scalar fields which behave as the perfect fluid, $\rho = \sum_{i=1}^{N}\dot{\phi}_i^{\,2}/2 + V$ and $p = \sum_{i=1}^{N}\dot{\phi}_i^{\,2}/2 - V$.

- Isocurvature/entropy perturbations ($\mathcal{S}$)
  Entropy perturbation is the perturbation by relative fluctuations between species in the universe which leave the total density unperturbed:

$$\frac{\delta\rho}{\rho} = 0. \tag{5.3}$$

  The entropy perturbation perturbs the path orthogonal to the background trajectory. It perturbs neither the total energy density, nor the expansion rate of the universe, thus the curvature is not perturbed by the *isocurvature* perturbation.

The isocurvature perturbation of the two species x and y is [24]

$$\mathcal{S}_{xy} \equiv \frac{\delta\rho_x}{\rho_x + p_x} - \frac{\delta\rho_y}{\rho_y + p_y}, \tag{5.4}$$

where $p_i = \omega_i \rho_i$. For having the adiabatic mode, one has $\mathcal{S}_{xy} = 0$.

In the case of the single field inflation, there is one degree of freedom giving the unique background trajectory. The possible fluctuation is the fluctuation along the trajectory, so only the curvature perturbation is obtained by the model.

5.2 Evolution of Multiple Scalar Fields

5.2.1 Background Equations

The Lagrangian density corresponding to this model is

$$\mathcal{L} = -\frac{1}{2}\sum_{i=1}^{N} g^{\mu\nu}\phi_{i,\mu}\phi_{i,\nu} - V(\phi_1, \cdots, \phi_N). \tag{5.5}$$

The two Friedmann equations for N scalar fields are

$$H^2 = \frac{1}{3m_{pl}^2}\left[V(\phi_1, \cdots, \phi_N) + \sum_{i=0}^{N}\frac{1}{2}\dot{\phi_i}^2\right], \tag{5.6}$$

$$\frac{\ddot{a}}{a} = \frac{1}{3m_{pl}^2}\left[V(\phi_1, \cdots, \phi_N) - \sum_{i=0}^{N}\dot{\phi_i}^2\right]. \tag{5.7}$$

The background homogeneous equations read

$$\ddot{\phi_i} + 3H\dot{\phi_i} + \frac{\partial V}{\partial \phi_i} = 0\,, \tag{5.8}$$

The slow roll parameters can be defined analogously to the single field case as.

$$\epsilon_i = -\frac{\dot{H}}{H^2} = 2m_{pl}^2\left(\frac{H_{\phi_i}}{H}\right)^2, \tag{5.9}$$

$$\epsilon_t = \sum_{i=1}^{N}\epsilon_i, \tag{5.10}$$

$$\eta_{ij} = m_{pl}^2\frac{V_{\phi_i\phi_j}}{V} = -\delta_1. \tag{5.11}$$

5.2.2 Perturbation Equations

The spacetime linearly perturbed about the FRW spacetime is considered in the longitudinal gauge as

$$ds^2 = -(1+2\psi)dt^2 + a^2(1-2\psi)\delta_{ij}dx^i dx^j \,. \tag{5.12}$$

The perturbation equations are derived from the perturbed Einstein equations:

$$\delta\ddot{\phi}_i + 3H\delta\dot{\phi}_i + \frac{k^2}{a^2}\delta\phi_i + \sum_j V_{\phi_i\phi_j}\delta\phi_j = -2V_{\phi_i}\psi + 4\dot{\phi}_i\dot{\psi}. \tag{5.13}$$

Moreover, the perturbed Einstein equations give the energy density and pressure constraints as [26]

$$3H\left(\dot{\psi} + H\psi\right) + \frac{k^2}{a^2}\psi = -\frac{1}{2m_{pl}^2}\delta\rho, \tag{5.14}$$

$$\dot{\psi} + H\psi = -\frac{1}{2m_{pl}^2}\delta p. \tag{5.15}$$

Here $\delta\rho$ is the total energy density perturbation and δp is the total pressure perturbation [26].

$$\delta\rho = \sum_i \left[\dot{\phi}_i\left(\delta\dot{\phi}_i - \dot{\phi}_i\psi\right) + V_{\phi_i}\delta\phi_i\right] \tag{5.16}$$

$$\delta p_{,l} = -\sum_i \dot{\phi}_i\delta\phi_{i,l} \,. \tag{5.17}$$

The simplest model of the multi-field inflation is the double inflation corresponding to two inflationary stages during the inflation epoch.

5.3 Double Inflation Formalism

There are two inflatons ϕ and χ, whose equations of motion for the homogeneous parts are

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0, \tag{5.18a}$$

$$\ddot{\chi} + 3H\dot{\chi} + V_\chi = 0. \tag{5.18b}$$

The scalar fields also have fluctuations, $\delta\phi$ and $\delta\chi$, so the equations of motion for their perturbation parts are

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} + \left(\frac{k^2}{a^2} + V_{\phi\phi}\right)\delta\phi = -2V_\phi\psi + 4\dot{\phi}\dot{\psi} - V_{\phi\chi}\delta\chi \,, \tag{5.19a}$$

$$\delta\ddot{\chi} + 3H\delta\dot{\chi} + \left(\frac{k^2}{a^2} + V_{\chi\chi}\right)\delta\chi = -2V_\chi\psi + 4\dot{\chi}\dot{\psi} - V_{\phi\chi}\delta\phi \,. \tag{5.19b}$$

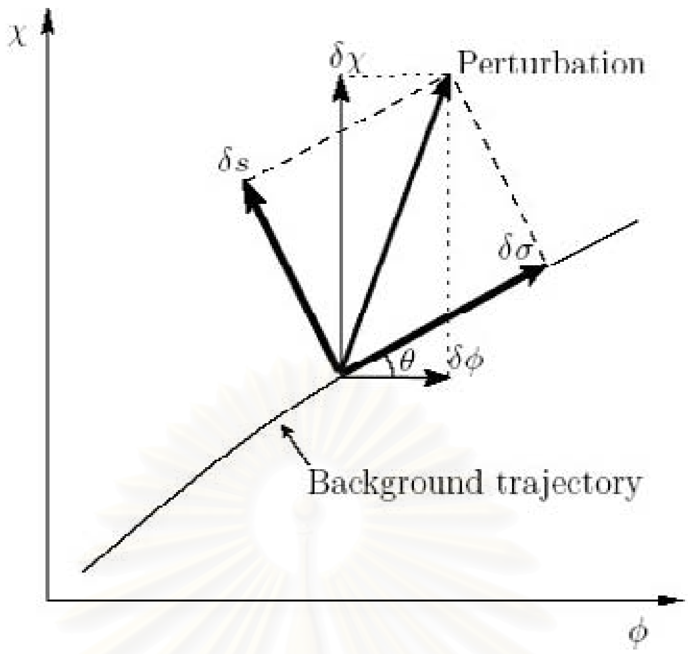

Figure 5.1: The perturbation is decomposed into an adiabatic ($\delta\sigma$) and entropy (δs) components [26].

The coupling between the two fields leads to correlations between adiabatic and entropy perturbations. In order to discuss this, one introduces the perturbation fields: the adiabatic field, σ, and the entropy field, s, for convenience. The decomposition into $\delta\sigma$ and δs of the two fields are characterized by the rotation in the field space as shown in Figure 5.1.

$$\delta\sigma \;=\; (\cos\theta)\delta\phi + (\sin\theta)\delta\chi, \quad \delta s \;=\; -(\sin\theta)\delta\phi + (\cos\theta)\delta\chi, \tag{5.20}$$

where θ is the angle between the trajectory and the ϕ-axis in the field space with

$$\cos\theta = \frac{\dot\phi}{\sqrt{\dot\phi^2 + \dot\chi^2}}, \qquad \sin\theta = \frac{\dot\chi}{\sqrt{\dot\phi^2 + \dot\chi^2}}. \tag{5.21}$$

The equations of motion for adiabatic and entropy field perturbations are

$$\delta\ddot\sigma + 3H\delta\dot\sigma + \left(\frac{k^2}{a^2} + V_{\sigma\sigma} - \dot\theta^2\right)\delta\sigma \;=\; -2V_\sigma\psi + 4\dot\sigma\dot\psi + 2\frac{d}{dt}(\dot\theta\delta s) - \frac{2V_\sigma}{\dot\sigma}\dot\theta\delta s, \tag{5.22}$$

$$\delta\ddot s + 3H\delta\dot s + \left(\frac{k^2}{a^2} + V_{ss} + 3\dot\theta^2\right)\delta s \;=\; 4m_{pl}^2\frac{\dot\theta}{\dot\sigma}\frac{k^2}{a^2}\psi, \tag{5.23}$$

where

$$V_{\sigma\sigma} = (\cos^2\theta)V_{\phi\phi} + (\sin 2\theta)V_{\phi\chi} + (\sin^2\theta)V_{\chi\chi}, \tag{5.24a}$$

$$V_{ss} = (\sin^2\theta)V_{\phi\phi} - (\sin 2\theta)V_{\phi\chi} + (\cos^2\theta)V_{\chi\chi}. \tag{5.24b}$$

The sources of the curvature can be obtained by using the equations (5.15) and (5.17). The solution of the gravitational potential is

$$\psi = \frac{1}{2am_{pl}^2}\int a\dot{\sigma}\delta\sigma dt\,. \tag{5.25}$$

As we discuss in the section 5.1, only the adiabatic field perturbation perturbs the curvature.

Because the adiabatic field perturbation is not gauge invariant, the gauge invariant variable called the Sasaki-Mukhanov [23] is introduced:

$$Q_i \equiv \delta\phi_i + \frac{\dot{\phi}_i}{H}\psi. \tag{5.26}$$

Now the gauge invariant perturbations are

$$Q_\sigma \;=\; (\cos\theta)Q_\phi + (\sin\theta)Q_\chi, \quad Q_s \;=\; (\cos\theta)Q_\chi - (\sin\theta)Q_\phi \;=\; \delta s. \tag{5.27}$$

It can be seen that the relative entropy perturbation is automatically gauge invariant. The equation for the adiabatic field perturbation is rewritten as

$$\ddot{Q}_\sigma + 3H\dot{Q}_\sigma + \left(\frac{k^2}{a^2} + V_{\sigma\sigma} - \dot{\theta}^2 - \frac{1}{m_{pl}^2a^3}\frac{d}{dt}\Big(\frac{a^3\dot{\sigma}^2}{H}\Big)\right)Q_\sigma =$$
$$2\frac{d}{dt}(\dot{\theta}\delta s) - \Big(\frac{V_\sigma}{\dot{\sigma}} + \frac{\dot{H}}{H}\Big)\dot{\theta}\delta s. \tag{5.28}$$

The right hand side of the equation shows that the relative entropy perturbation plays a role as an additional source of the adiabatic perturbation in the curved trajectory in the field space. Thus the two perturbations decouple when $\dot{\theta} = 0$.[3] However, there are no sources of the entropy perturbation on the large scale.

According to the slow roll conditions, the first terms of the equations (5.23) and (5.28) can be neglected. Besides, long wavelength limit gives $k^2/a^2 \ll 1$. The differential equations become

$$\delta\dot{s} + \frac{1}{3H}(V_{ss} + 3\dot{\theta}^2)\delta s \;=\; 0, \tag{5.29}$$

$$\dot{Q}_\sigma + \frac{1}{3H}\left(V_{\sigma\sigma} - \dot{\theta}^2 - \frac{1}{m_{pl}^2a^3}\frac{d}{dt}\Big(\frac{a^3\dot{\sigma}^2}{H}\Big)\right)Q_\sigma$$
$$= \frac{1}{3H}\left[2\frac{d}{dt}(\dot{\theta}\delta s) - \Big(\frac{V_\sigma}{\dot{\sigma}} + \frac{\dot{H}}{H}\Big)\dot{\theta}\delta s\right]. \tag{5.30}$$

[3] According to [26], $\dot{\theta} = -\frac{V_s}{\dot{\sigma}}$. It can be seen that $\dot{\sigma} \neq 0$ when $\dot{\theta} = 0$.

The general solutions are

$$\delta s \;=\; B(k)g(t), \quad Q_\sigma \;=\; A(k)f(t) + P(t). \tag{5.31}$$

Here $f(t)$ is the homogeneous solution while $P(t)$ is the particular solution. In the case of the single field inflation, the long wavelength solutions are evaluated at the horizon crossing because the curvature perturbation is frozen when it crosses outside the horizon. Thus considering quantum fluctuations at $k = aH_k$, the scale-dependent amplitudes are determined by

$$Q_i \approx \frac{H_k}{\sqrt{2k^3}} e_i(\mathbf{k}), \tag{5.32}$$

where H_k is the Hubble parameter at the horizon exit and $e_i(\mathbf{k})$ is the classical Gaussian random variable which satisfies the relation $\langle e_i(\mathbf{k})\rangle = 0$ and $\langle e_i(\mathbf{k})e_j^*(\mathbf{k}')\rangle = \delta_{ij}\delta^{(3)}(\mathbf{k}-\mathbf{k}')$ [26]. Therefore at the horizon exit

$$A = \frac{H_k}{\sqrt{2k^3}}\, e_Q(\mathbf{k})\,, \quad B = \frac{H_k}{\sqrt{2k^3}}\, e_s(\mathbf{k})\,, \tag{5.33a}$$

$$f \;=\; g \;=\; 1, \quad p \;=\; 0. \tag{5.33b}$$

However, in the double field model, Q_σ is not frozen in any scale because it couples with δs. Thus the values in (5.33) are set to be the initial conditions for the superhorizon scale. The solution for the entropy field perturbation is

$$g(t) = \exp\left[\int_{t_k}^{t} -\frac{\mu_s^2}{3H}\, dt,\right], \tag{5.34}$$

where t_k is the horizon exit time and $\mu_s^2 = V_{ss} + 3\dot{\beta}^2$ is the effective squared mass of the entropy field. The ratio of the effective squared mass and the squared Hubble parameter can be written in terms of the slow roll parameters in the lowest order as

$$-\frac{\mu_s^2}{3H^2} = \frac{-(\epsilon_\chi\eta_{\phi\phi} + \epsilon_\phi\eta_{\chi\chi}) + 2(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})\eta_{\phi\chi}}{\epsilon_t}. \tag{5.35}$$

The time-dependent part of $\mu_s^2/3H^2$ is the second order in the slow roll parameters which can be neglected, so the mass of δs is constant.
Using the definition of the e-folding, $g(t)$ becomes

$$\begin{aligned} g(t) \;&=\; \exp\left[\int_{t_k}^{t_f} -\frac{\mu_s^2}{3H^2}\, H dt \;+\; \int_{t_f}^{t} -\frac{\mu_s^2}{3H^2}\, H dt,\right] \\ &\approx\; \exp\left[-\frac{\mu_s^2}{3H^2}(N(t_k) - N(t))\right], \end{aligned} \tag{5.36}$$

where $N(t_k) \equiv N_k = \int_{t_k}^{t_f} H\ dt$ is the number of e-folding between the horizon exit time and the end of inflation[4].

The perturbations are correlated until the end of inflation. The solution at the end of inflation is

$$g(t_f) \;=\; \exp\left[\left(\frac{-(\epsilon_\chi \eta_{\phi\phi} + \epsilon_\phi \eta_{\chi\chi}) + 2(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})\eta_{\phi\chi}}{\epsilon_t}\right)_k N_k\right]. \tag{5.37}$$

The right hand side terms are evaluated at the horizon exit[5].

The homogeneous solution for the adiabatic perturbation is

$$\begin{aligned} f(t) &= \exp\left[\int_{t_k}^{t} -\frac{\mu_Q^2}{3H}\, dt,\right] \\ &\approx \exp\left[-\frac{\mu_Q^2}{3H^2}(N_k - N(t))\right], \end{aligned} \tag{5.38}$$

where $\mu_Q^2 \equiv \left(V_{\sigma\sigma} - \dot\theta^2 - \kappa^2 a^{-3}(a^3\dot\sigma^2/H)^\bullet\right)$ is the effective squared mass of the adiabatic field. The $\mu_Q^2/3H^2$ term to the lowest order in the slow roll parameters is

$$-\frac{\mu_Q^2}{3H^2} = \frac{-(\epsilon_\chi \eta_{\chi\chi} + \epsilon_\phi \eta_{\phi\phi}) - 2(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})\eta_{\phi\chi}}{\epsilon_t} + 2\epsilon_t\ . \tag{5.39}$$

The homogeneous adiabatic perturbation at the end of inflation is

$$f(t_f) = \exp\left[\left(-\frac{\epsilon_\chi \eta_{\chi\chi} + \epsilon_\phi \eta_{\phi\phi} + 2(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})\eta_{\phi\chi}}{\epsilon_t} + 2\epsilon_t\right)_k N_k\right]. \tag{5.40}$$

The source terms of the adiabatic perturbation can be written in terms of the entropy perturbation (in the lowest order of the slow roll parameters) as

$$\begin{aligned} b(t) &\equiv \frac{2}{3H}\left[(\dot\beta\delta s)^{\cdot} - \left(\frac{V_\sigma}{\dot\sigma} + \frac{\dot H}{H}\right)\dot\beta\delta s\right] \\ &= 2\dot\theta\delta s \\ &= 2H\left(\frac{\dot\theta}{H}\right)B(k)g(t), \end{aligned} \tag{5.41}$$

where the quantity $\dot\theta/H$ can be expressed as [24]

$$\frac{\dot\theta}{H} \approx \frac{1}{\epsilon_t}[(\epsilon_\chi - \epsilon_\phi)\eta_{\phi\chi} + (\eta_{\phi\phi} - \eta_{\chi\chi})(\pm\sqrt{\epsilon_\chi})(\pm\sqrt{\epsilon_\phi})], \tag{5.42}$$

---

[4] According to the recent observations, $N_k \approx 65$.

[5] $\frac{\mu_s^2}{3H^2}$ is constant up to the lowest order in the slow roll parameters therefore it can be evaluated at the horizon exit.

which takes the value at the horizon exit scale. The particular solution is

$$\begin{aligned}
P(t) &= \exp\left[\int_{t_k}^{t} -\frac{\mu_Q^2}{3H}\,dt,\right]\int_{t_k}^{t}\exp\left[\int_{t'_k}^{t'}\frac{\mu_Q^2}{3H}\,dt'',\right]b(t')dt' \\
&= 2B\left(\frac{\dot\theta}{H}\right)f(t)\int_{N_k}^{N(t)}\exp\left[\frac{\mu_s^2-\mu_Q^2}{3H^2}(N'(t)-N'_k)\right]dN' \\
P(t_f) &= 2B\left(\frac{\dot\theta}{H}\right)g(t_f)\,\frac{e^{CN_k-1}}{C}, \qquad (5.43)
\end{aligned}$$

where

$$\begin{aligned}
C &\equiv \frac{\mu_s^2-\mu_Q^2}{3H^2} \qquad (5.44)\\
&= \frac{(\epsilon_\phi-\epsilon_\chi)\eta_{\chi\chi}+(\epsilon_\chi-\epsilon_\phi)\eta_{\phi\phi}-4\,(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})\eta_{\phi\chi}}{\epsilon_t}+2\epsilon_t\,, \qquad (5.45)
\end{aligned}$$

is related to the difference in the effective squared mass of the two fields, and $g(t_f)=e^{-CN_k}f(t_f)$. The power spectrum of Q_σ is

$$\begin{aligned}
\mathcal{P}_{Q_\sigma} &= \frac{k^3}{2\pi^2}|Q_\sigma|^2 \\
&= \left(\frac{H_k}{2\pi}\right)^2\left[|f^2|+|\tilde P^2|\right]. \qquad (5.46)
\end{aligned}$$

Here, $P=B\tilde P$. In the same way, the other power spectra are

$$\mathcal{P}_{\delta s} = \left(\frac{H_k}{2\pi}\right)^2|g^2|, \qquad (5.47)$$

$$\mathcal{P}_{Q_\sigma\delta s} = \left(\frac{H_k}{2\pi}\right)^2 g\tilde P. \qquad (5.48)$$

As we know the adiabatic perturbation perturbs the curvature therefore the curvature perturbation is characterized by this mode. From the definition of $\mathcal{R}$ in (5.2). The curvature perturbation for the two inflatons are

$$\mathcal{R} = \psi + H\frac{\dot\phi\delta\phi+\dot\chi\delta\chi}{\dot\phi^2+\dot\chi^2}. \qquad (5.49)$$

Using (5.20), (5.21) and (5.28), the curvature perturbation is related to the adiabatic perturbation as

$$\mathcal{R} = \frac{H}{\dot\sigma}Q_\sigma\,. \qquad (5.50)$$

Here $\dot\sigma^2=\dot\phi^2+\dot\chi^2=2m_{pl}^2H^2\epsilon_t$. Thus the power spectrum of $\mathcal{R}$ is

$$\begin{aligned}
\mathcal{P}_{\mathcal{R}} &= \left(\frac{H(t_f)}{\dot\sigma(t_f)}\right)^2\mathcal{P}_{Q_\sigma} \\
&= \left(\frac{H_k}{2\pi m_{pl}}\right)^2\frac{1}{2\epsilon_t}\left[|f^2|+|\tilde P^2|\right], \qquad (5.51)
\end{aligned}$$

where $\dot{\phi}_i^{\,2} = 2m_{pl}^2 H^2 \epsilon_i$.

The total energy density perturbation in (5.16) can be separated in those of the two fields, $\delta\rho = \delta\rho_\phi + \delta\rho_\chi$. From the definition of the isocurvature perturbation in (5.4), one finds

$$\mathcal{S}_{\chi\phi} = \dot{\delta}_{\chi\phi} - 3H\delta_{\chi\phi}, \tag{5.52}$$

where $\delta_{\chi\phi} = \delta\chi/\dot{\chi} - \delta\phi/\dot{\phi}$. Its first derivative can be neglected because of the slow roll approximations. From the equation (5.27), the relative entropy field can be written in terms of the field perturbations as

$$\delta s = \frac{\dot{\phi}\delta\chi - \dot{\chi}\delta\phi}{\sqrt{\dot{\phi}^2 + \dot{\chi}^2}} \;=\; \frac{\dot{\phi}\dot{\chi}}{\sqrt{\dot{\phi}^2 + \dot{\chi}^2}}\,\delta_{\chi\phi}. \tag{5.53}$$

The first derivative of $\delta_{\chi\phi}$ can be neglected because of the slow roll approximations, thus the isocurvature perturbation can be written in terms of the entropy field perturbation as

$$\mathcal{S}_{\chi\phi} = -\frac{3\sqrt{\epsilon_t}}{\sqrt{2}m_{pl}^2(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})}\,\delta s. \tag{5.54}$$

The power spectrum for the isocurvature perturbation is

$$\begin{aligned}\mathcal{P}_{\mathcal{S}} &= \left(-\frac{3\sqrt{\epsilon_t}}{\sqrt{2}m_{pl}^2(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})}\right)^2 \mathcal{P}_{\delta s} \\ &= \frac{9}{2}\left(\frac{H_k}{2\pi m_{pl}}\right)^2 \frac{\epsilon_t}{\epsilon_\phi\epsilon_\chi}\,|g^2(t_f)|. \end{aligned} \tag{5.55}$$

The cross-spectrum between $\mathcal{R}$ and $\mathcal{S}$ is found to be

$$\begin{aligned}\mathcal{P}_{\mathcal{C}} &= \frac{H(t_f)}{\dot{\sigma}(t_f)}\left(-\frac{3\sqrt{\epsilon_t}}{\sqrt{2}m_{pl}^2(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})}\right)^2 \mathcal{P}_{Q_\sigma\delta s} \\ &= -3\frac{\dot{\theta}}{H}\,\frac{e^{CN_k-1}}{C}\left(\frac{H_k}{2\pi m_{pl}}\right)^2 \frac{|g^2(t_f)|}{(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})}. \end{aligned} \tag{5.56}$$

The spectral index is defined by

$$n - 1 \equiv \frac{d\,\mathrm{ln}\mathcal{P}}{d\,\mathrm{ln}k} = \frac{(1+\epsilon_t)}{\mathcal{P}}\frac{d\,\mathcal{P}}{d\,\mathrm{lna}}\bigg|_{k=aH}. \tag{5.57}$$

By the definition of the number of e-folds and $g(t_f) = e^{-CN_k} f(t_f)$, the spectral indices for all power spectra are

$$
\begin{aligned}
n_{\mathcal{R}} - 1 &= -6\epsilon_t + \frac{2(\epsilon_\phi \eta_{\phi\phi} + \epsilon_\chi \eta_{\chi\chi}) + 4(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})\eta_{\phi\chi}}{\epsilon_t} \\
&\quad - \frac{8|f^2(t_f)|}{|f^2(t_f)| + |P^2(t_f)|} \left(\frac{\dot{\theta}}{H}\right)^2 \frac{e^{-CN_k}}{C}(1 - e^{-CN_k}), \qquad (5.58) \\
n_{\mathcal{S}} - 1 &= -2\epsilon_t + \frac{2(\epsilon_\phi \eta_{\chi\chi} + \epsilon_\chi \eta_{\phi\phi}) - 4(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})\eta_{\phi\chi}}{\epsilon_t}, \qquad (5.59) \\
n_{\mathcal{C}} - 1 &= -2\epsilon_t + \frac{2(\epsilon_\phi \eta_{\chi\chi} + \epsilon_\chi \eta_{\phi\phi}) - 4(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})\eta_{\phi\chi}}{\epsilon_t} - \frac{Ce^{CN_k}}{e^{CN_k} - 1}. \qquad (5.60)
\end{aligned}
$$

The running of spectral indices can be expressed as

$$
\begin{aligned}
\frac{dn_{\mathcal{R}}}{d\ln k} &= -8\left(\frac{\dot{\theta}}{H}\right)^2 \frac{(e^{-CN_k} - 2e^{-2CN_k})}{1 + 4\left(\frac{\dot{\theta}}{H}\right)^2\left(\frac{1-e^{-CN_k}}{C}\right)^2} \\
&\quad -64\left(\frac{\dot{\theta}}{H}\right)^4 \frac{e^{-2CN_k}\left(\frac{1-e^{-CN_k}}{C}\right)^2}{\left(1 + 4\left(\frac{\dot{\theta}}{H}\right)^2\left(\frac{1-e^{-CN_k}}{C}\right)^2\right)^2}, \qquad (5.61) \\
\frac{dn_{\mathcal{S}}}{d\ln k} &= 0, \qquad (5.62) \\
\frac{dn_{\mathcal{C}}}{d\ln k} &= -\frac{C^2 e^{CN_k}}{e^{CN_k} - 1}. \qquad (5.63)
\end{aligned}
$$

It can be seen that the running of the isocurvature perturbation is zero up to the second order in the slow roll parameters. In contrast, if the adiabatic and the entropy fields have no equal effective squared mass, the running of the cross-spectrum is non-zero. Moreover, in the curved trajectory and $C \neq 0$, it is possible to have the non-zero running of the curvature perturbation.

5.4 Double Inflation with Supersymmetric Potential

Adding another field into the inflaton potential is a possible way to solve problems of the single field model. However, some explanation for the existence of the second field is needed. One possible theory is *Supersymmetry*. Supersymmetry is the symmetry between boson and fermion. Supersymmetry may play very important roles in cosmology such as inflation and the origin of the cold dark matter. Furthermore, the supersymmetric potential or *superpotential* is suitable for being the inflaton potential because its shape is consistent with the slow roll approximation and inflation can end. In addition, all particles have their own *superpartner* which has the same mass but different spin in the supersymmetric theory. Inflatons are scalar fields which are superpartners of the spin-$\frac{1}{2}$ fermions. So inflatons are fundamental particles in this theory.

One considers the superpotential including two superfields, which are the fields in supersymmetry, S and φ [28]

$$V = |k_0\varphi\bar{\varphi} - \mu^2|^2 + k_0^2|S|^2\big(|\varphi|^2 + |\bar{\varphi}|^2\big) + D\text{-terms}, \tag{5.64}$$

where $\bar{\varphi}$ is the complex conjugate of φ. k_0 and μ are positive constants. One finds the minimum of the superpotential at $\langle S\rangle = 0$, $\langle\varphi\bar{\varphi}\rangle = \langle\varphi\rangle\langle\bar{\varphi}\rangle = \frac{\mu^2}{k_0}$ and the condition for vanishing of the D-terms is $|\langle\varphi\rangle| = |\langle\bar{\varphi}\rangle|$ which is called *flat direction* [32]. This condition makes the minimum lies at $\langle S\rangle = 0$ and $\langle\varphi\rangle = \langle\bar{\varphi}\rangle^* = \pm\frac{\mu}{\sqrt{k_0}}$. So, the complex superfields can be replaced by real scalar fields ϕ and χ as

$$S = \tfrac{\phi}{\sqrt{2}}, \quad \varphi = \bar{\varphi} = \tfrac{\chi}{2}. \tag{5.65}$$

Substituting (5.65) in the potential, one obtains

$$V = \frac{k_0^2}{16}\left(\chi^2 - \frac{4\mu^2}{k_0}\right)^2 + \frac{1}{4}k_0^2\phi^2\chi^2. \tag{5.66}$$

There are no D-terms in the new potential because the condition of flat direction.

The hybrid potential consisting of two inflaton fields with the coupling constant g is

$$V = \frac{\lambda}{4}\left(\chi^2 - \frac{M^2}{\lambda}\right)^2 + \frac{1}{2}g^2\phi^2\chi^2 + \frac{1}{2}m^2\phi^2. \tag{5.67}$$

Comparing the supersymmetric potential with the hybrid potential, One finds that the mass term of the field ϕ is added into the superpotential. All constants are constrained by $\mu^2 = \frac{M^2}{2\sqrt{\lambda}}$ and $\frac{k_0^2}{2} = g^2 = 2\lambda$. So, supersymmetric hybrid inflation corresponds to the value of the coupling constant: $g = \sqrt{2\lambda}$.

Figure 5.2 shows the supersymmetric potential. There are two inflationary stages. The first stage begins when the inflaton rolls slowly along the nearly flat potential in the ϕ-direction along the $\chi = 0$-axis. This stage continues until $\phi = \phi_c \equiv \frac{M}{g}$, then the second stage occurs in the χ-direction along the $\phi = 0$-axis. Inflation ends when the inflaton reaches one of the true minima $\chi = \chi_f \equiv \pm\frac{M}{\sqrt{\lambda}}$.

According to the second consistency condition in (3.31), both of the inflationary stages occur when the effective masses of the two fields are lighter than

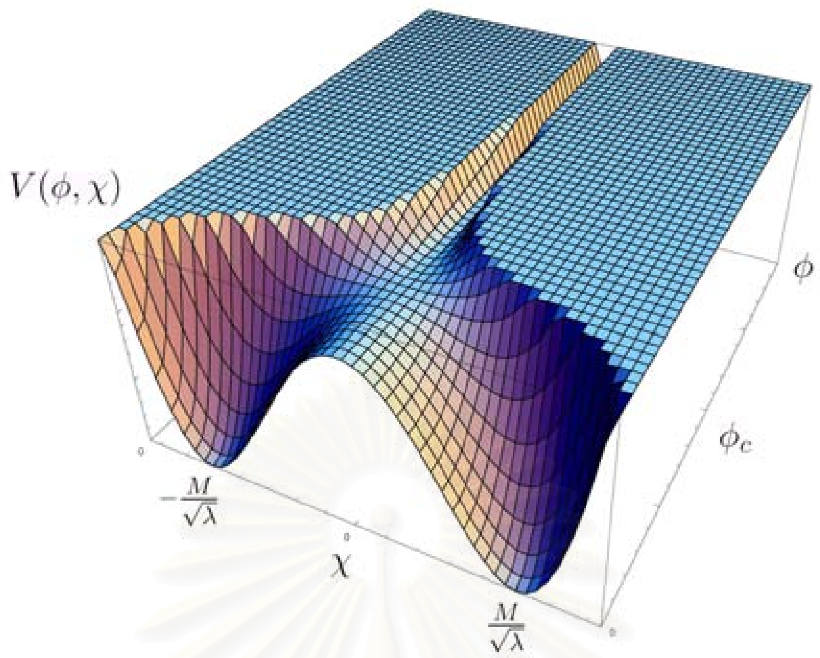

Figure 5.2: Supersymmetric potential [33].

the Hubble rate:

$$\begin{aligned} m_\phi^2 &\equiv m^2 \ll H^2, \\ m_\chi^2 &\equiv g^2\phi^2 - M^2 \ll H^2. \end{aligned}$$

The first stage of the inflation occurs when $\phi \geq \phi_c$ in the almost flat ϕ-direction along the $\chi = 0$-axis. The effective squared mass of χ is positive in this stage. The potential is approximately reduced into the form

$$V \simeq \frac{M^4}{4\lambda} + \frac{m^2\phi^2}{2}. \tag{5.68}$$

In the case of $M > m$, which gives $V \simeq \frac{M^4}{4\lambda}$, the Hubble rate is almost constant around $\phi \approx \phi_c$. [6] Using the first background Friedmann equation, the Hubble rate is

$$H \simeq H_c \equiv \frac{1}{2\sqrt{3\lambda}}\frac{M^2}{m_{pl}}. \tag{5.69}$$

Therefore the slow roll conditions are valid when

$$M^2 \gg 2\sqrt{3\lambda}\, m m_{pl}, \quad \text{and} \tag{5.70}$$

$$M^2 \gg m_{pl}^2\lambda. \tag{5.71}$$

[6] In the supersymmetric case, $M^2 = 2\lambda\phi_c^2$ at $\phi \approx \phi_c$. So $V = \frac{M^2\phi_c^2}{2} + \frac{m^2\phi_c^2}{2} \approx \frac{M^2\phi_c^2}{2}$.

The second inflationary stage occurs after the symmetry breaking due to the tachyonic[7] instability. The field evolves slowly in the almost flat χ-direction along $\phi = 0$-axis when $\phi < \phi_c$. The potential is approximately described as

$$V \simeq \frac{\lambda}{4}\left(\chi^2 - \frac{M^2}{\lambda}\right)^2. \tag{5.72}$$

The e-folds N in the two stages are respectively

$$N_1 \approx \frac{M^4}{4\lambda m^2 m_{pl}^2} \ln \frac{\phi_i}{\phi_c}, \tag{5.73}$$

$$N_2 \approx \frac{M^2}{4\lambda m_{pl}^2} \ln \frac{\chi_f}{\chi_c}, \tag{5.74}$$

where ϕ_i and ϕ_c are the values at the beginning and the end of the first stage. χ_c and χ_f are the values at the beginning and the end of the second stage respectively. Under the consistency conditions, one sees that the e-folds in both stages are greater than one. The total amount of e-folding is $N_t \equiv N_1 + N_2$ whose constrained value is closed to 65 according to CMB observations.
The *effective* masses at the minima of the potential are

$$\begin{aligned} m_{\phi_c} &= \frac{g}{\sqrt{\lambda}} M, \\ m_{\chi_f} &= \sqrt{2} M. \end{aligned}$$

Here, m_{ϕ_c} is the mass at $\phi = \phi_c = \pm M/g$ while m_{χ_f} is the mass at $\chi = \chi_f = \pm M/\sqrt{\lambda}$. Considering the supersymmetric case, one obtains the equal mass of the inflatons at the potential minima: $m_\phi = m_\chi = \sqrt{2} M$.

The slow roll parameters at the horizon exit scale are

$$\begin{aligned} \epsilon_\phi &= \frac{m^2}{6H_k^2}\left(\frac{m^2\phi^2}{V}\right), \quad \eta_{\phi\phi} = \frac{m^2}{3H_k^2} \\ \epsilon_\chi &= \frac{m_\chi^2}{6H_k^2}\left(\frac{m_\chi^2\chi^2}{V}\right), \quad \eta_{\chi\chi} = \frac{m_\chi^2}{3H_k^2} \\ \eta_{\phi\chi} &= 2m_{pl}^2 g^2 \frac{\phi\chi}{V}. \end{aligned} \tag{5.75}$$

Before the symmetry breaking, the slow roll conditions are valid in the ϕ-direction along the $\chi = 0$-axis. Therefore the ϵ_t comes mainly from ϵ_ϕ ($\epsilon_\chi \approx 0$). The spectral indices in the first stage are found to be

$$\begin{aligned} n_{\mathcal{R}} - 1 &\approx -6\epsilon_\phi + 2\eta_{\phi\phi} = \frac{2}{3}\frac{m^2}{H_k^2}\left(1 - \frac{3}{2}\frac{m^2\phi^2}{V}\right), \\ n_{\mathcal{S}} - 1 &\approx -2\epsilon_\phi + 2\eta_{\chi\chi} = \frac{2}{3}\frac{m_\chi^2}{H_k^2} - \frac{m^2}{H_k^2}\left(\frac{1}{3}\frac{m^2\phi^2}{V}\right), \\ n_{\mathcal{C}} - 1 &\approx -2\epsilon_\phi + 2\eta_{\chi\chi} - \frac{Ce^{CN_k}}{e^{CN_k} - 1}. \end{aligned}$$

[7]Tachyon is the particle whose squared mass is negative.

As we know C is related to the effective squared mass difference between the adiabatic and entropy fields, it can also be written in terms of the squared mass difference of the inflatons as

$$C \approx \eta_{\chi\chi} - \eta_{\phi\phi} = \frac{m_\chi^2}{3H_k^2} - \frac{m^2}{3H_k^2}.$$

Light field conditions give $C \to 0$. In the limit $|C|N_k \ll 1$, one obtains

$$\frac{Ce^{CN_k}}{e^{CN_k} - 1} = \frac{C(1 + CN_k + \frac{C^2N_k^2}{2} + ...)}{CN_k + \frac{C^2N_k^2}{2} + ...} \approx \frac{1}{N_k}. \quad (5.76)$$

When the condition $m \ll M$ is satisfied, the spectral indices generated in the first stage are

$$n_{\mathcal{R}} \approx 1 + \frac{2}{3}\frac{m^2}{H_k^2}, \quad (5.77)$$

$$n_{\mathcal{S}} \approx 1 + \frac{2}{3}\frac{m_\chi^2}{H_k^2}, \quad (5.78)$$

$$n_{\mathcal{C}} \approx n_{\mathcal{S}} - \frac{1}{N_k}. \quad (5.79)$$

The results show that there are blue-tilted spectra in both of curvature and isocurvature perturbations. They are consistent with the data at the cluster scale. For the correlation, $n_{\mathcal{C}}$ can be blue-tilted at the beginning of the first stage when $\phi \gg \phi_c$. At $\phi \approx \phi_c$, $m_\chi^2 \approx 0$ gives the red-tilted spectrum of the correlation.
Following the same procedure, the spectral indices in the second stage are

$$n_{\mathcal{R}} \approx 1 + \frac{2}{3}\frac{m_\chi^2}{H_k^2}, \quad (5.80)$$

$$n_{\mathcal{S}} \approx 1 + \frac{2}{3}\frac{m^2}{H_k^2}, \quad (5.81)$$

$$n_{\mathcal{C}} \approx n_{\mathcal{S}} - \frac{1}{N_k}. \quad (5.82)$$

The spectrum of the curvature perturbations is red-tilted due to the negative squared mass of the field χ while that of the isocurvature perturbations is still blue-tilted. The paper [28] considers only the limit $N_k \gg 1$, therefore $n_{\mathcal{C}} \approx n_{\mathcal{S}}$. In the case of the supersymmetric potential, only the blue-tilted spectra are possible in the first inflationary stage.

However, the red-tilted spectra can be generated, even in the first stage, by the double inflation model with other potentials. If one considers non-interacting inflatons with $g = 0$, one obtains $m_\chi^2 = -M^2$. So the red-tilted spectra can be produced in both stages when $M^2 > 0$:

$$n_{\mathcal{S}_{1^{st}}} = n_{\mathcal{R}_{2^{nd}}} \approx 1 - \frac{2}{3}\frac{M^2}{H_k^2}. \quad (5.83)$$

Let us come back to the supersymmetric case. The strength of the correlation can be obtained from the curvature of the trajectory in the field space. The curvature in the two stages are found to be

$$\frac{\dot{\theta}}{H}\Big|_{1st} \approx \eta_{\phi\chi} = 2m_{pl}^2 g^2 \frac{\phi\chi}{V}, \tag{5.84}$$
$$\frac{\dot{\theta}}{H}\Big|_{2nd} \approx -\eta_{\phi\chi}. \tag{5.85}$$

In the limit of flat potential, the field ϕ decreases to the critical value ϕ_c during the first stage. This leads to the decreasing of $\dot{\theta}/H$ which gives the weak correlation between the perturbations. In contrast, during the second stage $\dot{\theta}/H$ increases due to the increasing of the field χ when it evolves to a potential minimum. Thus there is the strong correlation between the adiabatic and entropy perturbations. These results are consistent with [28].

Only the large scale perturbations or the small k modes are considered. This case corresponds to the large N_k because the number of e-folds shows the enlarged wavelength of the mode k at the end of inflation relative to its initial size. The larger N_k, the longer wavelength of the perturbation. The large scale perturbations come from the quantum fluctuations generated at the beginning of the first stage and cross outside the horizon immediately. Therefore the long wavelength modes have $N_k \approx N_t$.

According to [30], $50 \leq N_{2nd} \leq 65$ (where the total amount of e-folding is $N_t \approx 65$) satisfies the conditions for the suppressed isocurvature perturbations at the end of inflation. Thus the second stage dominates inflation, $N_{2nd} \rightarrow N_t$. For the large scale perturbations, one finds $N_k \approx N_{2nd}$.

We use the numerical values of the model parameters in [28][8], the values are within the same limit as in [30], and are giving by

$$M = 4.26 \times 10^{-6} m_{pl}, \; m = 1.0 \times 10^{-6} m_{pl}, \; \chi_c = 5.0 \times 10^{-3} \chi_f. \tag{5.86}$$

Moreover, [30] gives the power spectrum of the correlation at $N_k \approx 60$:

$$\mathcal{P}_{\mathcal{C}} \sim 10^{-11}. \tag{5.87}$$

In order to obtain one of our parameters by using the $\mathcal{P}_{\mathcal{C}}$ in the above equation, we consider $N_{2nd} = 60$ with the total amount of e-folding $N_t = 65$ in this model. From the number of e-folds expressed in (5.74), the coupling constant of the two fields is constrained to be

$$\lambda = 4 \times 10^{-13}. \tag{5.88}$$

[8] One uses the values in the case (b) of Fig.11 where $M_p = \sqrt{8\pi} m_{pl}$ is the Planck mass.

The coupling constant and the mass of χ lead to the final value of χ as[9]

$$\chi_f = \frac{M}{\sqrt{\lambda}} = 6.74 m_{pl}. \quad (5.89)$$

The value of χ at the beginning of the second stage can be obtained by the numerical values in (5.86):

$$\chi_c = 5.0 \times 10^{-3} \chi_f = 0.03 m_{pl}. \quad (5.90)$$

Although inflation is dominated by the potential energy of χ, the evolution of ϕ is required because the horizon exit for the long wavelength modes occurs in the first stage. The obtained value of the field ϕ_c is

$$\phi_c = \frac{M}{\sqrt{2\lambda}} = 4.77 m_{pl}. \quad (5.91)$$

Here the number of e-folds for the first stage is $N_{1st} = 5$. The initial value of ϕ can be obtained by using the equations (5.73), (5.86) and (5.88) ,

$$\begin{aligned} \ln\left(\frac{\phi_i}{\phi_c}\right) &= 0.024 \\ \phi_i &= 1.03\phi_c = 4.89 m_{pl}. \end{aligned} \quad (5.92)$$

It can be seen that during the first stage of inflation, all fields are of the order of the Planck mass.

In the second stage ϵ_t comes mainly from ϵ_χ, thus the solutions for the adiabatic and entropy perturbations are

$$\begin{aligned} f(t_f) &= e^{-\eta_{\chi\chi} N_k}, \; g(t_f) = e^{-\eta_{\phi\phi} N_k}, \\ p(t_f) &= 2g(t_f)\frac{\dot{\theta}}{H}\frac{e^{CN_k}-1}{C} \approx 2g(t_f)\frac{\dot{\theta}}{H}N_k. \end{aligned} \quad (5.93)$$

The time-independent effective masses of the two perturbation fields allow us to evaluate the right hand side of (5.93) at the horizon exit scale. At this scale, the curvature of the field trajectory reads[10]

$$\left(\frac{\dot{\theta}}{H}\right)_k = -\frac{4}{3}\frac{\lambda\phi_k\chi_k}{H_k^2}. \quad (5.94)$$

Using the slow roll parameters from (5.76), the solutions for the two perturbations become

$$\begin{aligned} f(t_f) &= e^{-\frac{m_\chi^2}{3H_k^2}N_k}, \; g(t_f) = e^{-\frac{m^2}{3H_k^2}N_k}, \\ p(t_f) &= -\frac{8}{3}\frac{\lambda\phi_k\chi_k N_k}{H_k^2}\, e^{-\frac{m^2}{3H_k^2}N_k}. \end{aligned} \quad (5.95)$$

[9]Note that we use the positive values for all fields.

[10]The subscript k of any quantity denotes its value at the horizon exit scale.

The power spectra can be expressed as

$$\begin{aligned}
\mathcal{P}_{\mathcal{R}} &= 9\left(\frac{H_k}{2\pi}\right)^2 \frac{H_k^4}{m_\chi^4 \chi_k^2}\Bigg[e^{-2\frac{m_\chi^2}{3H_k^2}N_k} \\
&\quad + \left(\frac{8}{3}\right)^2 \left(\frac{\lambda \phi_k \chi_k N_k}{H_k^2}\right)^2 e^{-2\frac{m^2}{3H_k^2}N_k}\Bigg], \qquad (5.96)\\
\mathcal{P}_{\mathcal{S}} &= 81\left(\frac{H_k}{2\pi}\right)^2 \frac{H_k^4}{m^4 \phi_k^2}\, e^{-2\frac{m^2}{3H_k^2}N_k}, \qquad (5.97)\\
\mathcal{P}_{\mathcal{C}} &= 72\left(\frac{\lambda}{4\pi^2}\right) \frac{H_k^4}{m^2 m_\chi^2}\left(e^{-2\frac{m^2}{3H_k^2}N_k}\right) N_k. \qquad (5.98)
\end{aligned}$$

The long wavelength perturbations exit the horizon at the early time in the first stage, therefore the value of ϕ at the horizon exit is closed to its value at the beginning of this stage: $\phi_k \approx \phi_i = 4.89 m_{pl}$. The effective squared mass of the field χ at the horizon exit is

$$m_\chi^2 \approx 2\lambda\phi_i^2 - M^2 = 9.3 \times 10^{-13} m_{pl}^2. \qquad (5.99)$$

Besides, for large scale perturbations, $N_k \approx N_{2nd} = 60$. Substituting all inflationary parameters into (5.96), (5.97) and (5.98), then the solutions yield

$$\begin{aligned}
\mathcal{P}_{\mathcal{R}} &= \frac{2.64\times 10^{23}}{m_{pl}^4}\left(\frac{H_k^6}{\chi_k^2}\right)\left[e^{-\frac{3.75\times 10^{-11} m_{pl}^2}{H_k^2}} + \left(4.26\times 10^{-21}\right)\frac{\chi_k^2 m_{pl}^2}{H_k^4}\right] \qquad (5.100)\\
\mathcal{P}_{\mathcal{S}} &= \frac{8.58\times 10^{22}}{m_{pl}^6}\left(H_k^6\right) e^{-\frac{4\times 10^{-11} m_{pl}^2}{H_k^2}}, \qquad (5.101)\\
\mathcal{P}_{\mathcal{C}} &= \frac{2.35\times 10^{13}}{m_{pl}^4}\left(H_k^4\right) e^{-\frac{4\times 10^{-11} m_{pl}^2}{H_k^2}}. \qquad (5.102)
\end{aligned}$$

Note that the Hubble parameter is time-dependent. Using the large scale cross-spectrum in (5.87), the Hubble parameter at the exit time is

$$H_k = 2.83 \times 10^{-6} m_{pl}. \qquad (5.103)$$

We know from the Chapter IV that the inflation energy scale can be characterized by the Hubble parameter. The inflation energy scale obtained by the supersymmetric double inflation is of the order of $10^{-6} m_{pl}$.

Using (5.103), one obtains the isocurvature power spectrum:

$$\mathcal{P}_{\mathcal{S}} = 2.92 \times 10^{-13} \qquad (5.104)$$

which is the same order as its numerical value in [28].

In order to estimate the curvature power spectrum, the value of χ_k is required. Figure 5.2 shows that inflation begins with the small field value of χ. For long wavelength modes, they cross outside the horizon in the early time of the inflation period. Thus, χ_k must be less than its initial value in the second stage: $\chi_k < \chi_c = 0.03 m_{pl}$. Choosing $\chi_k = 0.023 m_{pl}$, the power spectrum of the curvature perturbation predicted by the supersymmetric double inflation model is

$$\mathcal{P}_{\mathcal{R}} = 2.5 \times 10^{-9}, \tag{5.105}$$

which is two and four orders of magnitude greater than $\mathcal{P}_{\mathcal{C}}$ and $\mathcal{P}_{\mathcal{S}}$ respectively. So the total power spectrum is dominated and approximately given by $\mathcal{P}_{\mathcal{R}}$. The power spectrum on the large scale as given by this model is of the same order as the observational results.
Note that the upper limit of $\chi_k = 0.03 m_{pl}$ gives $\mathcal{P}_{\mathcal{R}} = 1.45 \times 10^{-9}$.

We summarize that isocurvature perturbations in multi-field models are an additional source of curvature perturbations on the curved trajectory in the field space. Thus the conservation of the large scale amplitudes of the curvature perturbations as in the single field model is violated. However, the power spectrum of the isocurvature perturbations is constant on the large scale. The correlation between the two kinds of perturbations appears and become strong in the second inflationary stage. In addition, the existence of more than one field causes the origin of the cold dark matter. For double inflation model, the decay of the inflaton ϕ creates the ordinary matter whereas the inflaton χ decays into the cold dark matter. Furthermore, the slow roll conditions do not give the unnatural flat potential any longer because the multi-field inflation models can be motivated by particle physics. The double inflation with supersymmetric potential is in out interest. In this case, the obtained power spectrum $\mathcal{P} \approx \mathcal{P}_{\mathcal{R}} = 2.5 \times 10^{-9}$ is the same order as in WMAP data on the superhorizon scale. Moreover, the result is in good agreement with the power spectrum in the galactic scale, $\mathcal{P} = 2.46 \times 10^{-9}$. The spectrum of the curvature perturbations is blue-tilted in the first stage but red-tilted in the second stage which correspond to the results from [30]. The running of the spectral indices can be non-zero in this model.

CHAPTER VI

CONCLUSION

In the single-field inflation models, both commutative inflation and noncommutative inflation are discussed. In this case, there are no relative fluctuations because only a single field exists. The power spectrum is purely adiabatic. The obtained curvature power spectrum is constant on the superhorizon scale. For the commutative power-law inflation, the spectral index is red-tilted and the running is zero.

There are the minimum length scale caused by the quantum effect in the noncommutative inflation model. The spacetime noncommutativity presents as the extra terms in $\tilde{\mathcal{P}}_{\mathcal{R}}, \tilde{n}_{\mathcal{R}}$ and $\frac{d\tilde{n}_{\mathcal{R}}}{d\ln k}$. When $l_s \to 0$, the three quantities reduce to the commutative version.

The extra terms in the power spectrum make it suppressed on the superhorizon scale by the factor $(1+\mu)^{-4}$. This is closed to the low multipoles of the CMB power spectrum. The delay of the exit time due to the noncommutative effect also slightly shifts the value of the spectral index, and the blue-tilted spectrum presents. The non-zero running of the spectral index can appear by the same procedure. Note that its value is always negative as the large scale data. In our results, the string length in the power-law model is four orders of magnitude larger than the Planck scale, $l_s \sim 10^{-29}$ cm. The result corresponds to [15, 18]. In this case, inflation occurs when the size of the universe is about 10^{-24} cm. and the scale factor evolves with time as $a \propto t^{12.171}$.

For the multi-field inflation case, the light masses of all inflaton fields are necessary to have the inflationary stages. In the presence of more than one inflatons, there exists of two kinds of the primordial perturbations. First, the curvature perturbations which effect the gravitational potential (as shown in (5.25)), as well as the expansion of the universe. Their large scale amplitudes are not conserved because of the existence of the latter kind, the isocurvature perturbations. The isocurvature perturbations are the perturbations by the relative fluctuations among species in the universe. The strength of the correlation between the two perturbations depends on the curvature of the field trajectory.

In the double inflation model, the first stage occurs in the ϕ-direction along the $\chi = 0$-axis with its initial value is large. This stage stops when the field reaches the critical value, then the second stage begins in the χ-direction along the $\phi = 0$-axis. Inflation ends when the field χ reaches one of its minima. The energy lost of the two fields may create ordinary matter and dark matter in the universe. During the second inflationary stage, in the case of the negative squared mass, inflation is driven by the symmetry breaking due to the tachyonic instability.

As shown in the equation (3.135), the single field inflation model usually gives the red-tilted spectrum when the slow roll limits are employed [7]. In contrast, the double inflation model can give both red-tilted and blue-tilted spectra depending on the coupling constant between the two fields.

For the supersymmetric version of inflation, the potential is realistic even if the slow roll conditions are valid. The power spectrum is 2.5×10^{-9} which is consistent with the large scale data of WMAP. The contribution of the isocurvature perturbations to the CMB power spectrum is required to be small compared to the curvature perturbations. In our results, $\mathcal{P}_{\mathcal{S}}/\mathcal{P}_{\mathcal{R}} \sim 10^{-4}$, as well as the correlations, $\mathcal{P}_{\mathcal{C}}/\mathcal{P}_{\mathcal{R}} \sim 10^{-2}$. The results correspond to [30] with $\mathcal{P}_{\mathcal{S}}/\mathcal{P}_{\mathcal{R}} < 0.004$ and $\mathcal{P}_{\mathcal{C}}/\mathcal{P}_{\mathcal{R}} < 0.07$. Moreover, in the limit of large N_k (long wavelength limit), all spectra in the first stage are blue-tilted. In the second stage the spectra of $\mathcal{S}$ and $\mathcal{C}$ are still blue-tilted whereas the $\mathcal{R}$'s becomes red-tilted. For the running of the spectral indices, the isocurvature perturbations have the zero running while the others do not.

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APPENDICES

APPENDIX A

LIMITS FOR HAVING INFLATION

A.1 Equation of Motion in Conformal Time Coordinate

From

$$\frac{1}{\sqrt{-g}}\partial_\nu(g^{\mu\nu}\sqrt{-g}\partial_\mu\phi) - V_\phi = 0, \tag{A.1}$$

the equation of motion is

$$\begin{aligned}
\frac{1}{a^4}\Big[\partial_\nu\Big(\frac{1}{a^2}\eta^{\mu\nu}a^4\Big)\partial_\mu\phi + \frac{1}{a^2}\eta^{\mu\nu}a^4\partial_\nu\partial_\mu\phi\Big] - V_\phi(\phi) &= 0\\
\frac{1}{a^4}\Big[\frac{\partial}{\partial\eta}(a^2)\eta^{\mu 0}\partial_\mu\phi + a^2(\eta^{00}\partial_0\partial_0\phi + \eta^{ij}\partial_i\partial_j\phi)\Big] - V_\phi(\phi) &= 0\\
\frac{1}{a^4}\Big[-2a\frac{da}{d\eta}\frac{\partial\phi}{\partial\eta} + a^2\Big(-\frac{\partial^2\phi}{\partial\eta^2} + \nabla^2\phi\Big)\Big] - V_\phi(\phi) &= 0\\
\frac{\partial^2\phi}{\partial\eta^2} + \frac{2}{a}\Big(\frac{da}{d\eta}\Big)\frac{\partial\phi}{\partial\eta} - \nabla^2\phi + a^2V_\phi(\phi) &= 0. \qquad (A.2)
\end{aligned}$$

A.2 The E-Folding

By the definition of the e-folding, we can rewrite it in term of the inflaton potential

$$\begin{aligned}
N &= \int_{t_i}^{t_f} H(t)dt \quad = \int_{\phi_i}^{\phi_f} d\phi H(\phi)\frac{dt}{d\phi}\\
&= \int_{\phi_i}^{\phi_f} \frac{H(\phi)}{\dot{\phi}}\,d\phi \quad = -\int_{\phi_i}^{\phi_f} \frac{3H^2(\phi)}{V_\phi(\phi)}\,d\phi\\
\therefore N &= \int_{\phi_f}^{\phi_i} \frac{1}{m_{pl}^2}\frac{V(\phi)}{V_\phi(\phi)}\,d\phi. \qquad (A.3)
\end{aligned}$$

Inflation needs more than 55 e-folds in order to solve all problems from big bang model.

A.3 Derivatives of the Potential

From

$$
\begin{aligned}
V(\phi) &= 2m_{pl}^4\Big[\frac{3}{2m_{pl}^2}H^2(\phi) - (H_\phi(\phi))^2\Big], \\
V_\phi(\phi) &= 2m_{pl}^4\Big[\frac{3}{m_{pl}^2}H(\phi)H_\phi(\phi) - 2H_\phi(\phi)H_{\phi\phi}(\phi)\Big] \\
&= 2m_{pl}^4\Big(\frac{3H(\phi)H_\phi(\phi)}{m_{pl}^2}\Big)\Big[1-\frac{2}{3}\frac{H_{\phi\phi}(\phi)}{H(\phi)}\Big] \\
&= 3\sqrt{2}m_{pl}H^2(\phi)\Big(\sqrt{2}m_{pl}\frac{H_\phi(\phi)}{H(\phi)}\Big)\Big[1-\frac{2}{3}\frac{H_{\phi\phi}(\phi)}{H(\phi)}\Big] \\
&\equiv -3\sqrt{2}m_{pl}H^2(\phi)\sqrt{\epsilon(\phi)}\big[1+\delta_1(\phi)\big],
\end{aligned} \tag{A.4}
$$

where $\delta_1 \equiv -2m_{pl}\frac{H_{\phi\phi}(\phi)}{H(\phi)} = \frac{\ddot{\phi}}{H}$.

From

$$
\begin{aligned}
V_\phi(\phi) &= 2m_{pl}^4\Big[\frac{3}{m_{pl}^2}H(\phi)H_\phi(\phi) - 2H_\phi(\phi)H_{\phi\phi}(\phi)\Big] \\
V_{\phi\phi}(\phi) &= 2m_{pl}^4\Big[\frac{3}{m_{pl}^2}\Big\{(H_\phi(\phi))^2 + H(\phi)H_{\phi\phi}(\phi)\Big\} \\
&\quad -2(H_{\phi\phi}(\phi))^2 - 2H_\phi(\phi)H_{\phi\phi\phi}(\phi)\Big] \\
&= H^2\Big[3(2m_{pl}^2)\big(\frac{H_\phi}{H}\big)^2 - 3\big(-2m_{pl}^2\frac{H_{\phi\phi}}{H}\big) - \big(2m_{pl}^2\frac{H_\phi}{H}\big)^2 \\
&\quad -\big(2m_{pl}^2\frac{H_\phi}{H}\big)\big(2m_{pl}^2\frac{H_{\phi\phi\phi}}{H}\big)\Big] \\
&= H^2\Big[3\epsilon - 3\delta_1 - \delta_1^2 - \epsilon\Big(2m_{pl}^2\big(\frac{H_{\phi\phi\phi}}{H}\big)^2\Big)\Big].
\end{aligned} \tag{A.5}
$$

From

$$H_\phi(\phi) = -\frac{1}{2m_{pl}^2}\dot{\phi}, \tag{A.6}$$

$$H_{\phi\phi\phi}(\phi) = -\frac{1}{2m_{pl}^2}\frac{d^3\phi}{dt^3}. \tag{A.7}$$

Consider

$$
\begin{aligned}
2m_{pl}^2\big(\frac{H_{\phi\phi\phi}}{H}\big)^2 &= 2m_{pl}^2H_{\phi\phi\phi}\big(\frac{H^2}{2m_{pl}^2{H_\phi}^2}\big)\big(2m_{pl}^2\frac{H_\phi}{H^2}\big) \\
&= -\frac{d^3\phi}{dt^3}\Big(\frac{1}{\epsilon}\Big)\Big(\frac{-\dot{\phi}}{H^2}\Big) \\
&\equiv \frac{\delta_2}{\epsilon},
\end{aligned} \tag{A.8}
$$

where $\delta_2 \equiv \frac{1}{H^2}\frac{d^3\phi}{dt^3}$. So,

$$V_{\phi\phi}(\phi) \;=\; 3H^2\Big[\epsilon(\phi) - \delta_1(\phi) - \frac{1}{3}\delta_1^2(\phi) - \frac{1}{3}\dot{\phi}\delta_2(\phi)\Big]. \tag{A.9}$$

Inflation occurs when $V_\phi \ll V$ and $V_{\phi\phi} \ll V$. These are the consistency conditions in Chapter III.

A.4 Calculation of η, $\frac{1}{a}\Big(\frac{d^2a}{d\eta^2}\Big)$ and $\frac{\ddot{a}}{a}$

From the definition of the conformal time,

$$\eta \;=\; \int \frac{d\eta}{da} \;=\; \int \frac{1}{aH^2}\, da$$

Integrating by parts to have

$$\eta \;=\; -\frac{1}{aH} + \int \frac{1}{a}\frac{d}{da}\Big(\frac{1}{H}\Big) \tag{A.10}$$

Consider $\frac{d}{da}(\frac{1}{H})$

$$\begin{aligned}\frac{d}{da}\Big(\frac{1}{H}\Big) &= \frac{-1}{H^2}\frac{dH}{da} = \frac{-1}{H^2}\frac{\partial H}{\partial\phi}\frac{\partial\phi}{\partial\tau}\frac{d\eta}{da}\\ &= \frac{-1}{H^2}\frac{d\eta}{da}(-2m_{pl}^2 aH_\phi)H_\phi\\ &= \frac{\epsilon}{aH}.\end{aligned}$$

The conformal time can be calculated as

$$\eta \;=\; -\frac{1}{aH} + \int \frac{\epsilon}{a^2H}\, da. \tag{A.11}$$

The value of the slow roll parameter must be much less than one during inflation, so ϵ can be treated as a constant. The first order expression is

$$\begin{aligned}\eta &\approx -\frac{1}{aH} + \epsilon\int\frac{1}{a^2H}\,da\\ &= -\frac{1}{aH} + \epsilon\Big(\int\frac{1}{a^2H} + \int\frac{1}{a}\frac{d}{da}\Big(\frac{1}{H}\Big)\Big)\\ &= -\frac{1}{aH} + \epsilon\Big(-\frac{1}{aH} + \int\epsilon\frac{1}{a^2H}\,da\Big)\\ &= -\frac{1}{aH} + \epsilon\Big[-\frac{1}{aH} + \epsilon\Big(-\frac{1}{aH} + \int\epsilon\frac{1}{a^2H}\,da\Big)\Big]\\ &= -\frac{1}{aH}(1+\epsilon+\epsilon^2+\epsilon^3+...)\\ \therefore \eta &\approx -\frac{1}{aH}\frac{1}{(1-\epsilon)}.\end{aligned} \tag{A.12}$$

Compute $\frac{1}{a}\left(\frac{d^2a}{d\eta^2}\right)$ by considering the Friedmann equations:

$$\frac{1}{a^2}\left(\frac{da}{d\eta}\right)^2 = \frac{1}{3m_{pl}^2}\Big[a^2V(\phi)+\Big(\frac{\partial\phi}{\partial\eta}\Big)^2\Big], \tag{A.13}$$

$$\frac{1}{a}\left(\frac{d^2a}{d\eta^2}\right)-\frac{1}{a^2}\left(\frac{da}{d\eta}\right)^2 = \frac{1}{3m_{pl}^2}\Big[a^2V(\phi)-\Big(\frac{\partial\phi}{\partial\eta}\Big)^2\Big]. \tag{A.14}$$

2(A.13)+(A.14) gives

$$\frac{1}{a}\left(\frac{d^2a}{d\eta^2}\right)+\frac{1}{a^2}\left(\frac{da}{d\eta}\right)^2 = \frac{a^2}{m_{pl}^2}V(\phi). \tag{A.15}$$

Substituting

$$V(\phi) = 3m_{pl}^2H^2(\phi)\Big[1-\frac{1}{3}\epsilon(\phi)\Big], \tag{A.16}$$

one obtains

$$\begin{aligned}\frac{1}{a}\left(\frac{d^2a}{d\eta^2}\right)+a^2H^2 &= 3a^2H^2\Big[1-\frac{1}{3}\epsilon(\phi)\Big]\\ \therefore\frac{1}{a}\left(\frac{d^2a}{d\eta^2}\right) &= a^2H^2(2-\epsilon).\end{aligned} \tag{A.17}$$

One calculates a^2H^2 by using (A.12), and substitutes in (A.17)

$$\begin{aligned}\frac{1}{a}\left(\frac{d^2a}{d\eta^2}\right) &= \frac{2-\epsilon}{\eta^2(1-\epsilon)^2}\\ &= \frac{1}{\eta^2}(2-\epsilon)(1+2\epsilon+...)\\ &= \frac{1}{\eta^2}(2+4\epsilon-\epsilon-2\epsilon^2+...)\\ \therefore\frac{1}{a}\left(\frac{d^2a}{d\eta^2}\right) &\approx \frac{1}{\eta^2}(2+3\epsilon).\end{aligned} \tag{A.18}$$

One finds $\frac{\ddot{a}}{a}$ by considering

$$\begin{aligned}\ddot{a} &= \frac{d}{dt}\Big(\frac{da}{dt}\Big) \quad = \frac{d}{dt}\Big(\frac{1}{a}\frac{da}{d\eta}\Big)\\ &= \frac{d}{d\eta}\Big(\frac{1}{a}\frac{da}{d\eta}\Big)\frac{d\eta}{dt} \quad = \frac{1}{a}\Big(\frac{d^2a}{d\eta^2}\frac{1}{a}-\frac{1}{a^2}\Big(\frac{da}{d\eta}\Big)^2\Big)\\ &= \frac{1}{a^2}\frac{d^2a}{d\eta^2}-\frac{1}{a^3}\Big(\frac{da}{d\eta}\Big)^2\end{aligned} \tag{A.19}$$

Using (A.17), the result is

$$\begin{aligned}\frac{\ddot{a}}{a} &= H^2(2-\epsilon)-H^2\\ &= H^2(1-\epsilon).\end{aligned} \tag{A.20}$$

A.5 Limit of Pressure and Energy Density During Inflation

$$
\begin{aligned}
\frac{d\ln H}{d\ln a} &= \frac{\frac{1}{H}dH/d\eta}{\frac{1}{a}da/d\eta} &= \frac{\frac{1}{aH}dH/d\eta}{\frac{1}{a^2}da/d\eta} \\
&= \frac{1}{aH^2}\frac{dH}{d\eta} &= \frac{1}{aH^2}\frac{\partial H}{\partial\phi}\frac{d\phi}{d\eta} \\
&= \frac{H_\phi}{aH^2}\frac{d\phi}{d\eta}. && \text{(A.21)}
\end{aligned}
$$

Consider $\frac{d\phi}{d\eta}$, from

$$
H_\phi(\phi) = -\frac{\dot{\phi}}{2m_{pl}^2} = -\frac{1}{2m_{pl}^2}\left(\frac{1}{a}\frac{d\phi}{d\eta}\right). \tag{A.22}
$$

Due to $\sqrt{\epsilon} = -\sqrt{2}m_{pl}\frac{H_\phi}{H}$, hence,

$$
\begin{aligned}
\frac{d\phi}{d\eta} &= 2m_{pl}^2 aH_\phi(\phi) \\
&= \sqrt{2}m_{pl}aH(\phi)\sqrt{\epsilon(\phi)} && \text{(A.23)}
\end{aligned}
$$

Substitute (A.23) into (A.21), the result is

$$
\begin{aligned}
\frac{d\ln H}{d\ln a} &= \frac{H_\phi}{aH^2}\sqrt{2}m_{pl}aH(\phi)\sqrt{\epsilon(\phi)} \\
&= -\epsilon, && \text{(A.24)}
\end{aligned}
$$

so the variation of the Hubble parameter with respect to the scale factor is the first order in the slow roll parameter.

From the equation of state $p = \omega\rho$, the association between the energy density and the scale factor is calculated by using the continuity equation:

$$
\rho \propto a^{-3(1+\omega)} \tag{A.25}
$$

$$
\frac{d\ln\rho}{d\ln a} \sim -3(1+\omega) \tag{A.26}
$$

From the first Friedmann equation in flat space, one has

$$
a \propto \eta^{\frac{2}{1+3\omega}} \tag{A.27}
$$

Using the (A.27), the Hubble parameter can be written in terms of the scale factor as follow,

$$
\begin{aligned}
H &= \frac{1}{a^2}\frac{da}{d\eta} = \frac{2}{a^2(1+3\omega)}\eta^{\frac{1-3\omega}{1+3\omega}} \\
&\propto a^{\frac{-3}{2}(1+\omega)}. && \text{(A.28)}
\end{aligned}
$$

From (A.25) and (A.28), one obtains;

$$\frac{d\ln H}{d\ln a} = \frac{1}{2}\frac{d\ln\rho}{d\ln a}. \tag{A.29}$$

In order to know the limit of the parameters during the period of inflation, this relation is used in the continuity equation

$$\begin{aligned}\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) &= 0 \\ \dot{\rho}a^3 + 3\rho\dot{a}a^2 + 3p\dot{a}a^2 &= 0 \\ \frac{d}{dt}(\rho a^3) &= -p\frac{d}{dt}(a^3),\end{aligned} \tag{A.30}$$

this is the first law of Thermodynamics $dU = dQ - dW, dW = PdV$ with $dQ = 0$ (adiabatic process).
Using $\frac{1}{x^3}dx^3 = 3d\ln x$, we find

$$\begin{aligned}p &= -\frac{d}{da^3}(\rho a^3) \quad = \quad -\Big[\rho + a^3\frac{d\rho}{da^3}\Big] \\ &= -\rho\Big[1 + \frac{\frac{1}{\rho}d\rho}{\frac{1}{a^3}da^3}\Big] \\ \therefore p &= -\rho\Big[1 + \frac{1}{3}\frac{d\ln\rho}{d\ln a}\Big].\end{aligned} \tag{A.31}$$

Now one considers how fast of the decreasing of the universe during inflation by using the second Friedmann equation,$\ddot{a} \propto -(\rho + 3p)$. Because $\ddot{a}$ must be positive, the negative pressure , $p < -\frac{\rho}{3}$ is the result. So,

$$\begin{aligned}p \quad = -\rho\Big[1 + \frac{1}{3}\frac{d\ln\rho}{d\ln a}\Big] &< -\frac{\rho}{3} \\ -\frac{1}{3}\frac{d\ln\rho}{d\ln a} &< \frac{2}{3}\end{aligned}$$

We know that the energy density must not be increasing as the universe is expanding. The derivative of ρ must be negative.

$$\Big|\frac{d\ln\rho}{d\ln a}\Big| < 2. \tag{A.32}$$

By substituting the value of the pressure in each epoch, one evaluates

$$\Big|\frac{d\ln\rho_m}{d\ln a}\Big| = 3, \tag{A.33}$$

$$\Big|\frac{d\ln\rho_r}{d\ln a}\Big| = 4, \tag{A.34}$$

where ρ_m and ρ_r are the energy density of the universe during the *matter (non-relativistic particle) - dominated* epoch and the *radiation (relativistic particle) - dominated* epoch respectively. We have seen that the energy density of the universe

in the inflation period decreases slowly compared with the others periods.
Let's consider the limit of the pressure during inflation. Replacing (A.29) in (A.31), so that

$$p = -\rho\Big[1 + \frac{2}{3}\frac{d\ln H}{d\ln a}\Big]. \tag{A.35}$$

By using (A.24), the pressure of the inflaton can be rewritten as

$$p = -\rho\Big[1 - \frac{2}{3}\epsilon\Big]. \tag{A.36}$$

During inflation, $\epsilon \in [0, 1)$, so the lowest limit of p is $p = -\rho$. This occurs when $\epsilon = 0$ which is the case of the cosmological constant, Λ. The uppermost limit is $p = -\frac{\rho}{3}$ occurring when $\epsilon = 1$.

APPENDIX B

PERTURBED

EINSTEIN EQUATION

B.1 Perturbed Christoffel Connections

From

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}\, g^{\alpha\rho}\left(g_{\rho\gamma,\beta} + g_{\beta\rho,\gamma} - g_{\beta\gamma,\rho}\right),$$

and

$$\begin{aligned}\delta\Gamma^{\alpha}_{\beta\gamma} &= \frac{1}{2}\,\delta g^{\alpha\rho}\left(g_{\rho\gamma,\beta} + g_{\beta\rho,\gamma} - g_{\beta\gamma,\rho}\right)\\ &\quad + \frac{1}{2}\, g^{\alpha\rho}\left(\delta g_{\rho\gamma,\beta} + \delta g_{\beta\rho,\gamma} - \delta g_{\beta\gamma,\rho}\right).\end{aligned}$$

Due to the diagonal metric structure, the Christoffel connections are symmetric between the two indices down, $\Gamma^{\alpha}_{[\mu\nu]} = 0$, This is called *torsion free.*
One finds the components of the unperturbed and perturbed Christoffel connections

$$\begin{aligned}\Gamma^{0}_{00} &= \frac{1}{2}\, g^{0\rho}\left(g_{\rho0,0} + g_{0\rho,0} - g_{00,\rho}\right)\\ &= \frac{1}{2}\, g^{00}\left(g_{00,0}\right)\\ &= \frac{1}{2}\left(\frac{-1}{a^2}\right)\left(\frac{d}{d\eta}\left(-a^2\right)\right)\\ \therefore\, \Gamma^{0}_{00} &= \frac{a'}{a}. \qquad \text{(B.1)}\\ \delta\Gamma^{0}_{00} &= \frac{1}{2}\,\delta g^{0\rho}\left(g_{\rho0,0} + g_{0\rho,0} - g_{00,\rho}\right) + \frac{1}{2}\, g^{0\rho}\left(\delta g_{\rho0,0} + \delta g_{0\rho,0} - \delta g_{00,\rho}\right)\\ &= \frac{1}{2}\,\delta g^{00}\left(g_{00,0}\right) + \frac{1}{2}\, g^{00}\left(\delta g_{00,0}\right)\\ &= \frac{1}{2}\left(\frac{2A}{a^2}\right)\left(-2aa'\right) + \frac{1}{2}\left(\frac{-1}{a^2}\right)\left(\frac{\partial}{\partial\eta}\left(-2Aa^2\right)\right)\\ \therefore\, \delta\Gamma^{0}_{00} &= A'. \qquad \text{(B.2)}\end{aligned}$$

$$\begin{aligned}
\Gamma^0_{0i} &= \frac{1}{2}\, g^{0\rho}\left(g_{\rho i},_0 + g_{0\rho},_i - g_{0i},_\rho\right)\\
&= \frac{1}{2}\, g^{00}\left(g_{0i},_0 + g_{00},_i - g_{0i},_0\right)\\
&= \frac{1}{2}\Big(\frac{-1}{a^2}\Big)\left(\partial_i\big(-a^2\big)\right)\\
\therefore \Gamma^0_{0i} &= 0. &\text{(B.3)}\\
\delta\Gamma^0_{0i} &= \frac{1}{2}\,\delta g^{0\rho}\left(g_{\rho i},_0 + g_{0\rho},_i - g_{0i},_\rho\right) + \frac{1}{2}\, g^{0\rho}\left(\delta g_{\rho i},_0 + \delta g_{0\rho},_i - \delta g_{0i},_\rho\right)\\
&= \frac{1}{2}\,\delta g^{00}\left(g_{00},_i\right) + \frac{1}{2}\, g^{00}\left(\delta g_{00},_i\right)\\
&= \frac{1}{2}\left(\frac{2A}{a^2}(0)\right) + \frac{1}{2}\Big(\frac{-1}{a^2}\Big)\left(\partial_i\big(-2Aa^2\big)\right)\\
\therefore \delta\Gamma^0_{0i} &= \partial_i A. &\text{(B.4)}
\end{aligned}$$

$$\begin{aligned}
\Gamma^i_{00} &= \frac{1}{2}\, g^{i\rho}\left(g_{\rho 0},_0 + g_{0\rho},_0 - g_{00},_\rho\right)\\
&= \frac{1}{2}\left(\frac{1}{a^2}\delta^{il}\right)\left(-g_{00},_l\right)\\
\therefore \Gamma^i_{00} &= 0. &\text{(B.5)}\\
\delta\Gamma^i_{00} &= \frac{1}{2}\,\delta g^{i\rho}\left(g_{\rho 0},_0 + g_{0\rho},_0 - g_{00},_\rho\right) + \frac{1}{2}\, g^{i\rho}\left(\delta g_{\rho 0},_0 + \delta g_{0\rho},_0 - \delta g_{00},_\rho\right)\\
&= \frac{1}{2}\,\delta g^{il}\left(-g_{00},_l\right) + \frac{1}{2}\, g^{il}\left(-\delta g_{00},_l\right)\\
&= 0 + \frac{1}{2}\left(\frac{1}{a^2}\delta^{il}\right)\Big(-\partial_l\big(-2Aa^2\big)\Big)\\
\therefore \delta\Gamma^i_{00} &= \partial^i A, &\text{(B.6)}
\end{aligned}$$

where $l = 1, 2, 3$.

$$\begin{aligned}
\Gamma^0_{ij} &= \frac{1}{2}\, g^{0\rho}\left(g_{\rho j},_i + g_{i\rho},_j - g_{ij},_\rho\right)\\
&= \frac{1}{2}\Big(\frac{-1}{a^2}\Big)\left(-\frac{\partial}{\partial\eta}\big(a^2\delta_{ij}\big)\right)\\
\therefore \Gamma^0_{ij} &= \frac{a'}{a}\delta_{ij}. &\text{(B.7)}\\
\delta\Gamma^0_{ij} &= \frac{1}{2}\,\delta g^{0\rho}\left(g_{\rho j},_i + g_{i\rho},_j - g_{ij},_\rho\right) + \frac{1}{2}\, g^{0\rho}\left(\delta g_{\rho j},_i + \delta g_{i\rho},_j - \delta g_{ij},_\rho\right)\\
&= \frac{1}{2}\left(\frac{2A}{a^2}\right)\big(-2aa'\delta_{ij}\big) + \frac{1}{2}\Big(\frac{-1}{a^2}\Big)\left(-\frac{\partial}{\partial\eta}\big(-2a^2\psi\delta_{ij}\big)\right)\\
&= -2\frac{a'}{a}A\delta_{ij} - \frac{1}{a^2}\left(2aa'\psi\delta_{ij} + a^2\psi'\delta_{ij}\right)\\
\therefore \delta\Gamma^0_{ij} &= -2\frac{a'}{a}(A+\psi)\delta_{ij} - \psi'\delta_{ij}. &\text{(B.8)}
\end{aligned}$$

$$
\begin{aligned}
\Gamma^i_{0j} &= \frac{1}{2}\, g^{i\rho}\left(g_{\rho j},_0 + g_{0\rho},_j - g_{0j},_\rho\right)\\
&= \frac{1}{2}\, g^{il}\left(g_{lj},_0 + g_{0l},_j - g_{0j},_l\right)\\
&= \frac{1}{2}\Big(\frac{1}{a^2}\delta^{il}\Big)\big(2aa'\delta_{lj}\big)\\
\therefore \Gamma^i_{0j} &= \frac{a'}{a}\delta^i{}_j. && \text{(B.9)}\\
\delta\Gamma^i_{0j} &= \frac{1}{2}\,\delta g^{i\rho}\left(g_{\rho j},_0 + g_{0\rho},_j - g_{0j},_\rho\right) + \frac{1}{2}\, g^{i\rho}\left(\delta g_{\rho j},_0 + \delta g_{0\rho},_j - \delta g_{0j},_\rho\right)\\
&= \frac{1}{2}\delta g^{il}\left(g_{lj},_0\right) + \frac{1}{2} g^{il}\left(\delta g_{lj},_0\right)\\
&= \frac{1}{2}\left(\frac{2\psi}{a^2}\delta^{il}\right)\Big(\frac{\partial}{\partial\eta}a^2\delta_{lj}\Big) + \frac{1}{2}\Big(\frac{1}{a^2}\delta^{il}\Big)\frac{\partial}{\partial\eta}\left(-2a^2\psi\delta_{lj}\right)\\
&= 2\frac{a'}{a}\psi\delta^i{}_j - 2\frac{a'}{a}\psi\delta^i{}_j - \psi'\delta^i{}_j\\
\therefore \delta\Gamma^i_{0j} &= \psi'\delta^i{}_j. && \text{(B.10)}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^i_{jk} &= \frac{1}{2}\, g^{i\rho}\left(g_{\rho k},_j + g_{j\rho},_k - g_{jk},_\rho\right)\\
&= \frac{1}{2}\, g^{il}\left(g_{lk},_j + g_{jl},_k - g_{jk},_l\right)\\
&= \frac{1}{2}\Big(\frac{1}{a^2}\delta^{il}\Big)\Big((\partial_j a^2)\delta_{lk} + \big(\partial_k a^2\big)\delta_{jl} - \big(\partial_l a^2\big)\delta_{jk}\Big)\\
\therefore \Gamma^i_{jk} &= 0. && \text{(B.11)}\\
\delta\Gamma^i_{jk} &= \frac{1}{2}\,\delta g^{i\rho}\left(g_{\rho k},_j + g_{j\rho},_k - g_{jk},_\rho\right) + \frac{1}{2}\, g^{i\rho}\left(\delta g_{\rho k},_j + \delta g_{j\rho},_k - \delta g_{jk},_\rho\right)\\
&= \frac{1}{2}\,\delta g^{ij}\left(g_{lk},_j + g_{jl},_k - g_{jk},_l\right) + \frac{1}{2}\, g^{il}\left(\delta g_{lk},_j + \delta g_{jl},_k - \delta g_{jk},_l\right)\\
&= 0 + \frac{1}{2}\left(\frac{1}{a^2}\delta^{il}\right)\Big(\partial_j\big(-2a^2\psi\big)\delta_{lk} + \partial_k\big(-2a^2\psi\big)\delta_{jl} - \partial_l\big(-2a^2\psi\big)\delta_{jk}\Big)\\
&= -\delta^{il}\left(\partial_j\psi\delta_{lk} + \partial_k\psi\delta_{jl} - \partial_l\psi\delta_{jk}\right)\\
\therefore \delta\Gamma^i_{jk} &= -\partial_j\psi\delta^i{}_k - \partial_k\psi\delta^i{}_j + \partial^i\psi\delta_{jk}. && \text{(B.12)}
\end{aligned}
$$

B.2 Perturbed Ricci Tensor

From

$$R_{\mu\nu} = \partial_\alpha\Gamma^\alpha_{\mu\nu} - \partial_\mu\Gamma^\alpha_{\nu\alpha} + \Gamma^\alpha_{\sigma\alpha}\,\Gamma^\sigma_{\mu\nu} - \Gamma^\alpha_{\sigma\nu}\,\Gamma^\sigma_{\mu\alpha}\,.$$

and

$$
\begin{aligned}
\delta R_{\mu\nu} &= \partial_\alpha\delta\Gamma^\alpha_{\mu\nu} - \partial_\mu\delta\Gamma^\alpha_{\nu\alpha} + \delta\Gamma^\alpha_{\sigma\alpha}\,\Gamma^\sigma_{\mu\nu} + \Gamma^\alpha_{\sigma\alpha}\,\delta\Gamma^\sigma_{\mu\nu}\\
&\quad - \delta\Gamma^\alpha_{\sigma\nu}\,\Gamma^\sigma_{\mu\alpha} - \Gamma^\alpha_{\sigma\nu}\,\delta\Gamma^\sigma_{\mu\alpha}.
\end{aligned}
$$

Like the Christoffel connections, the Ricci tensor is symmetric between the two indices down.

The unperturbed Ricci tensor are

$$
\begin{aligned}
R_{00} &= \partial_\alpha \Gamma^\alpha_{00} - \partial_0 \Gamma^\alpha_{0\alpha} + \Gamma^\alpha_{\sigma\alpha}\Gamma^\sigma_{00} - \Gamma^\alpha_{\sigma 0}\Gamma^\sigma_{0\alpha} \\
&= \partial_0 \Gamma^0_{00} + \partial_i \Gamma^i_{00} - \partial_0 \Gamma^0_{00} - \partial_0 \Gamma^i_{0i} + \Gamma^0_{00}\Gamma^0_{00} + \Gamma^0_{i0}\Gamma^i_{00} + \Gamma^i_{0i}\Gamma^0_{00} \\
&\quad + \Gamma^i_{ji}\Gamma^j_{00} - \Gamma^0_{00}\Gamma^0_{00} - \Gamma^0_{i0}\Gamma^i_{00} - \Gamma^i_{00}\Gamma^0_{0i} - \Gamma^i_{j0}\Gamma^j_{0i} \\
&= -\frac{\partial}{\partial\eta}\left(\frac{a'}{a}\right)\delta^i{}_i + 3\left(\frac{a'}{a}\right)^2 - \left(\frac{a'}{a}\right)^2 \delta^i j \delta^j i \\
\therefore R_{00} &= -3\,\frac{a''}{a} + 3\left(\frac{a'}{a}\right)^2. && \text{(B.13)} \\
R_{0i} &= \partial_\alpha \Gamma^\alpha_{0i} - \partial_0 \Gamma^\alpha_{i\alpha} + \Gamma^\alpha_{\sigma\alpha}\Gamma^\sigma_{0i} - \Gamma^\alpha_{\sigma i}\Gamma^\sigma_{0\alpha} \\
&= \partial_0 \Gamma^0_{0i} + \partial_j \Gamma^j_{0i} - \partial_0 \Gamma^0_{i0} - \partial_0 \Gamma^j_{ij} + \Gamma^0_{00}\Gamma^0_{0i} + \Gamma^0_{j0}\Gamma^j_{0i} + \Gamma^j_{0j}\Gamma^0_{0i} \\
&\quad + \Gamma^j_{kj}\Gamma^k_{0i} - \Gamma^0_{0i}\Gamma^0_{00} - \Gamma^0_{ji}\Gamma^j_{00} - \Gamma^j_{0i}\Gamma^0_{0j} - \Gamma^j_{ki}\Gamma^k_{0j} \\
&= \partial_j \left(\frac{a'}{a}\right)\delta^j{}_i \\
\therefore R_{0i} &= 0. && \text{(B.14)} \\
R_{ij} &= \partial_\alpha \Gamma^\alpha_{ij} - \partial_i \Gamma^\alpha_{j\alpha} + \Gamma^\alpha_{\sigma\alpha}\Gamma^\sigma_{ij} - \Gamma^\alpha_{\sigma j}\Gamma^\sigma_{i\alpha} \\
&= \partial_0 \Gamma^0_{ij} + \partial_k \Gamma^k_{ij} - \partial_i \Gamma^0_{j0} - \partial_i \Gamma^k_{jk} + \Gamma^0_{00}\Gamma^0_{ij} + \Gamma^0_{k0}\Gamma^k_{ij} + \Gamma^k_{0k}\Gamma^0_{ij} \\
&\quad + \Gamma^k_{lk}\Gamma^l_{ij} - \Gamma^0_{0j}\Gamma^0_{i0} - \Gamma^0_{kj}\Gamma^k_{i0} - \Gamma^k_{0j}\Gamma^0_{ik} - \Gamma^k_{lj}\Gamma^l_{ik} \\
&= \frac{\partial}{\partial\eta}\left(\frac{a'}{a}\right)\delta_{ij} + \left(\frac{a'}{a}\right)^2 \delta_{ij} + 3\left(\frac{a'}{a}\right)^2 \delta_{ij} - \left(\frac{a'}{a}\right)^2 \delta_{kj}\delta^k{}_i \\
&\quad - \left(\frac{a'}{a}\right)^2 \delta^k{}_j \delta_{ik} \\
\therefore R_{ij} &= \left(\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right)\delta_{ij}. && \text{(B.15)}
\end{aligned}
$$

One finds the 00-component of the perturbed Ricci tensor

$$
\begin{aligned}
\delta R_{00} &= \partial_\alpha \delta\Gamma^\alpha_{00} - \partial_0 \delta\Gamma^\alpha_{0\alpha} + \delta\Gamma^\alpha_{\sigma\alpha}\Gamma^\sigma_{00} + \Gamma^\alpha_{\sigma\alpha}\delta\Gamma^\sigma_{00} \\
&\quad - \delta\Gamma^\alpha_{\sigma 0}\Gamma^\sigma_{0\alpha} - \Gamma^\alpha_{\sigma 0}\delta\Gamma^\sigma_{0\alpha},
\end{aligned}
$$

All six components can be computed as

$$
\begin{aligned}
\partial_\alpha \delta\Gamma^\alpha_{00} &= \partial_0 \delta\Gamma^0_{00} + \partial_i \delta\Gamma^i_{00} \\
&= A'' + \partial_i \partial^i A. &\text{(B.16)}\\
\partial_0 \delta\Gamma^\alpha_{0\alpha} &= \partial_0 \delta\Gamma^0_{00} + \partial_0 \delta\Gamma^i_{0i} \\
&= A'' - 3\psi''. &\text{(B.17)}\\
\delta\Gamma^\alpha_{\sigma\alpha}\,\Gamma^\sigma_{00} &= \delta\Gamma^0_{00}\,\Gamma^0_{00} + \delta\Gamma^0_{i0}\,\Gamma^i_{00} + \delta\Gamma^i_{0i}\,\Gamma^0_{00} + \delta\Gamma^i_{ji}\,\Gamma^j_{00} \\
&= \frac{a'}{a}A' - 3\frac{a'}{a}\psi'. &\text{(B.18)}\\
\Gamma^\alpha_{\sigma\alpha}\,\delta\Gamma^\sigma_{00} &= \Gamma^0_{00}\,\delta\Gamma^0_{00} + \Gamma^0_{i0}\,\delta\Gamma^i_{00} + \Gamma^i_{0i}\,\delta\Gamma^0_{00} + \Gamma^i_{ji}\,\delta\Gamma^j_{00} \\
&= 4\frac{a'}{a}A'. &\text{(B.19)}\\
\delta\Gamma^\alpha_{\sigma 0}\,\Gamma^\sigma_{0\alpha} &= \delta\Gamma^0_{00}\,\Gamma^0_{00} + \delta\Gamma^0_{i0}\,\Gamma^i_{00} + \delta\Gamma^i_{00}\,\Gamma^0_{0i} + \delta\Gamma^i_{j0}\,\Gamma^j_{0i} \\
&= \frac{a'}{a}A' - \psi'\delta^i j\frac{a'}{a} \quad = \quad \frac{a'}{a}A' - 3\frac{a'}{a}\psi'. &\text{(B.20)}\\
\Gamma^\alpha_{\sigma 0}\,\delta\Gamma^\sigma_{0\alpha} &= \Gamma^0_{00}\,\delta\Gamma^0_{00} + \Gamma^0_{i0}\,\delta\Gamma^i_{00} + \Gamma^i_{00}\,\delta\Gamma^0_{0i} + \Gamma^i_{j0}\,\delta\Gamma^j_{0i} \\
&= \frac{a'}{a}A' - 3\frac{a'}{a}\psi'. &\text{(B.21)}
\end{aligned}
$$

So

$$
\delta R_{00} = \partial_i \partial^i A + 3\psi'' + 3\frac{a'}{a}\psi' + 3\frac{a'}{a}A'. \tag{B.22}
$$

The 0i-component of the Ricci tensor is

$$
\begin{aligned}
\delta R_{0i} &= \partial_\alpha \delta\Gamma^\alpha_{0i} - \partial_0 \delta\Gamma^\alpha_{i\alpha} + \delta\Gamma^\alpha_{\sigma\alpha}\,\Gamma^\sigma_{0i} + \Gamma^\alpha_{\sigma\alpha}\,\delta\Gamma^\sigma_{0i} \\
&\quad - \delta\Gamma^\alpha_{\sigma i}\,\Gamma^\sigma_{0\alpha} - \Gamma^\alpha_{\sigma i}\,\delta\Gamma^\sigma_{0\alpha}\,, &\text{(B.23)}
\end{aligned}
$$

Its six components are

$$
\begin{aligned}
\partial_\alpha \delta\Gamma^\alpha_{0i} &= \partial_0 \delta\Gamma^0_{0i} + \partial_j \delta\Gamma^j_{0i} \\
&= \frac{\partial}{\partial\eta}(\partial_i A) - \partial_j(-\psi')\delta^j{}_i \quad = \quad \partial_i A' - \partial_i \psi'. && \text{(B.24)} \\
\partial_0 \delta\Gamma^\alpha_{i\alpha} &= \partial_0 \delta\Gamma^0_{i0} + \partial_0 \delta\Gamma^j_{ij} \\
&= \partial_i A' + \frac{\partial}{\partial\eta}\left(-\partial_i\psi\delta^j{}_j - \partial_j\psi\delta^j{}_i + \partial^i\psi\delta_{ij}\right) \\
&= \partial_i A' - 3\partial_i\psi'. && \text{(B.25)} \\
\delta\Gamma^\alpha_{\sigma\alpha}\,\Gamma^\sigma_{0i} &= \delta\Gamma^0_{00}\,\Gamma^0_{0i}\delta\Gamma^0_{j0}\,\Gamma^j_{0i} + \delta\Gamma^j_{0j}\,\Gamma^0_{0i} + \delta\Gamma^j_{kj}\,\Gamma^k_{0i} \\
&= (\partial_j A)\frac{a'}{a}\delta^j{}_i + \left(-\partial_k\psi\delta^j{}_j - \partial_j\psi\delta^j{}_k + \partial^j\psi\delta_{kj}\right)\frac{a'}{a}\delta^k{}_i \\
&= \frac{a'}{a}\partial_i A - 3\frac{a'}{a}\partial_i\psi. && \text{(B.26)} \\
\Gamma^\alpha_{\sigma\alpha}\,\delta\Gamma^\sigma_{0i} &= \Gamma^0_{00}\,\delta\Gamma^0_{0i} + \Gamma^0_{j0}\,\delta\Gamma^j_{0i} + \Gamma^j_{0j}\,\delta\Gamma^0_{0i} + \Gamma^j_{kj}\,\delta\Gamma^k_{0i} \\
&= 4\frac{a'}{a}\partial_i A. && \text{(B.27)} \\
\delta\Gamma^\alpha_{\sigma i}\,\Gamma^\sigma_{0\alpha} &= \delta\Gamma^0_{0i}\,\Gamma^0_{00} + \delta\Gamma^0_{ji}\,\Gamma^j_{00} + \delta\Gamma^j_{0i}\,\Gamma^0_{0j} + \delta\Gamma^j_{ki}\,\Gamma^k_{0j} \\
&= \frac{a'}{a}\partial_i A + \left(-\partial_i\psi\delta^j{}_k - \partial_k\psi\delta^j{}_i + \partial^j\psi\delta_{ki}\right)\frac{a'}{a}\delta^k{}_j \\
&= \frac{a'}{a}\partial_i A - 3\frac{a'}{a}\partial_i\psi. && \text{(B.28)} \\
\Gamma^\alpha_{\sigma i}\,\delta\Gamma^\sigma_{0\alpha} &= \Gamma^0_{0i}\,\delta\Gamma^0_{00} + \Gamma^0_{ji}\,\delta\Gamma^j_{00} + \Gamma^j_{0i}\,\delta\Gamma^0_{0j} + \Gamma^j_{ki}\,\delta\Gamma^k_{0j} \\
&= \frac{a'}{a}\partial_j A\delta^j{}_i + \frac{a'}{a}\partial^j A\delta_{ji} \quad = \quad 2\frac{a'}{a}\partial_i A. && \text{(B.29)}
\end{aligned}
$$

So

$$
\delta R_{0i} = 2\partial_i\psi' + 2\frac{a'}{a}\partial_i A. \qquad \text{(B.30)}
$$

The ij-component of the Ricci tensor is

$$
\begin{aligned}
\delta R_{ij} &= \partial_\alpha\delta\Gamma^\alpha_{ij} - \partial_i\delta\Gamma^\alpha_{j\alpha} + \delta\Gamma^\alpha_{\sigma\alpha}\,\Gamma^\sigma_{ij} + \Gamma^\alpha_{\sigma\alpha}\,\delta\Gamma^\sigma_{ij} \\
&\quad - \delta\Gamma^\alpha_{\sigma j}\,\Gamma^\sigma_{i\alpha} - \Gamma^\alpha_{\sigma j}\,\delta\Gamma^\sigma_{i\alpha}, && \text{(B.31)}
\end{aligned}
$$

Its six components are

$$
\begin{aligned}
\partial_\alpha \delta\Gamma^\alpha_{ij} &= \partial_0 \delta\Gamma^0_{ij} + \partial_k \delta\Gamma^k_{ij} \\
&= \frac{\partial}{\partial\eta}\left(-2\frac{a'}{a}(A+\psi)\delta_{ij} - \psi'\delta_{ij}\right) + \partial_k\left(-\partial_i\psi\delta^k{}_j - \partial_j\psi\delta^k{}_i + \partial^k\psi\delta_{ij}\right) \\
&= -2\left(\frac{a''}{a} - \left(\frac{a'}{a}\right)^2\right)(A+\psi)\delta_{ij} - 2\frac{a'}{a}(A'+\psi')\delta_{ij} - \psi''\delta_{ij} \\
&\quad +\partial_k\partial^k\psi\delta_{ij} - 2\partial_i\partial_j\psi. \qquad \text{(B.32)}
\end{aligned}
$$

$$
\begin{aligned}
\partial_i \delta\Gamma^\alpha_{j\alpha} &= \partial_i \delta\Gamma^0_{j0} + \partial_i \delta\Gamma^k_{jk} \\
&= \partial_i\partial_j A + \partial_i\left(-\partial_j\psi\delta^k{}_k - \partial_k\psi\delta^k{}_j + \partial^k\psi\delta_{jk}\right) \\
&= \partial_i\partial_j A - 3\partial_i\partial_j\psi. \qquad \text{(B.33)}
\end{aligned}
$$

$$
\begin{aligned}
\delta\Gamma^\alpha_{\sigma\alpha}\,\Gamma^\sigma_{ij} &= \delta\Gamma^0_{00}\,\Gamma^0_{ij} + \delta\Gamma^0_{k0}\,\Gamma^k_{ij} + \delta\Gamma^k_{0k}\,\Gamma^0_{ij} + \delta\Gamma^l_{kl}\,\Gamma^k_{ij} \\
&= \frac{a'}{a}A'\delta_{ij} - 3\frac{a'}{a}\psi'\delta_{ij}. \qquad \text{(B.34)}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^\alpha_{\sigma\alpha}\,\delta\Gamma^\sigma_{ij} &= \Gamma^0_{00}\,\delta\Gamma^0_{ij} + \Gamma^0_{k0}\,\delta\Gamma^k_{ij} + \Gamma^k_{0k}\,\delta\Gamma^0_{ij} + \Gamma^l_{kl}\,\delta\Gamma^k_{ij} \\
&= \frac{a'}{a}\left(-2\frac{a'}{a}(A+\psi)\delta_{ij} - \psi''\delta_{ij}\right) - \frac{a'}{a}\delta^k{}_k\left(2\frac{a'}{a}(A+\psi)\delta_{ij} + \psi''\delta_{ij}\right) \\
&= -8\left(\frac{a'}{a}\right)^2(A+\psi)\delta_{ij} - 4\frac{a'}{a}\psi'\delta_{ij}. \qquad \text{(B.35)}
\end{aligned}
$$

$$
\begin{aligned}
\delta\Gamma^\alpha_{\sigma j}\,\Gamma^\sigma_{i\alpha} &= \delta\Gamma^0_{0j}\,\Gamma^0_{i0} + \delta\Gamma^0_{kj}\,\Gamma^k_{i0} + \delta\Gamma^k_{0j}\,\Gamma^0_{ik} + \delta\Gamma^l_{kj}\,\Gamma^k_{il} \\
&= -\frac{a'}{a}\psi'\delta_{ij} - \left(2\left(\frac{a'}{a}\right)^2(A+\psi)\delta_{ij} + \frac{a'}{a}\psi'\delta_{ij}\right). \qquad \text{(B.36)}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^\alpha_{\sigma j}\,\delta\Gamma^\sigma_{i\alpha} &= \Gamma^0_{0j}\,\delta\Gamma^0_{i0} + \Gamma^0_{kj}\,\delta\Gamma^k_{i0} + \Gamma^k_{0j}\,\delta\Gamma^0_{ik} + \Gamma^l_{kj}\,\delta\Gamma^k_{il} \\
&= -2\left(\frac{a'}{a}\right)^2(A+\psi)\delta_{ij} - 2\frac{a'}{a}\psi'\delta_{ij}. \qquad \text{(B.37)}
\end{aligned}
$$

So

$$
\begin{aligned}
\delta R_{ij} = &\left(-\frac{a'}{a}A' - 5\frac{a'}{a}\psi' - 2\frac{a''}{a}A - 2\left(\frac{a'}{a}\right)^2 A - 2\frac{a''}{a}\psi\right. \\
&\left. -2\left(\frac{a'}{a}\right)^2\psi - \psi'' + \partial_k\partial^k\psi\right)\delta_{ij} + \partial_i\partial_j\psi - \partial_i\partial_j A. \qquad \text{(B.38)}
\end{aligned}
$$

B.3 Perturbed Ricci Scalar

From

$$R = g^{\mu\alpha}R_{\alpha\mu}.$$

The unperturbed part is

$$\begin{aligned}
R &= g^{0\alpha} R_{\alpha 0} + g^{i\alpha} R_{\alpha i} \quad = \quad g^{00} R_{00} + g^{ij} R_{ji} \\
&= -\frac{1}{a^2}\left(-3\,\frac{a''}{a} + 3\left(\frac{a'}{a}\right)^2\right) + \frac{1}{a^2}\delta^{ij}\left(\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right)\delta_{ji} \\
\therefore R &= 6\frac{a''}{a^3}. \qquad \text{(B.39)}
\end{aligned}$$

The perturbed Ricci scalar is

$$\begin{aligned}
\delta R &= \delta g^{\mu\alpha} R_{\alpha\mu} + g^{\mu\alpha}\,\delta R_{\alpha\mu}. \\
&= \delta g^{00} R_{00} + \delta g^{ij} R_{ji} + g^{00}\,\delta R_{00} + g^{ij}\,\delta R_{ji} \\
&= \left(\frac{2A}{a^2}\right)\left(-3\,\frac{a''}{a} + 3\left(\frac{a'}{a}\right)^2\right) + \left(\frac{2\psi}{a^2}\delta^{ij}\right)\left(\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right)\delta_{ji} \\
&\quad + \left(\frac{-1}{a^2}\right)\left(\partial_i\partial^i A + 3\psi'' + 3\frac{a'}{a}\psi' + 3\frac{a'}{a}A'\right) + \frac{1}{a^2}\delta^{ij}\Bigg[-\frac{a'}{a}A' \\
&\quad - 5\frac{a'}{a}\psi' - 2\frac{a''}{a}A - 2\left(\frac{a'}{a}\right)^2 A - 2\frac{a''}{a}\psi - 2\left(\frac{a'}{a}\right)^2\psi - \psi'' \\
&\quad + \partial_k\partial^k\psi\Bigg]\delta_{ji} + \frac{1}{a^2}\delta^{ij}\left(\partial_i\partial_j\psi - \partial_i\partial_j A\right) \\
&= \left(\frac{1}{a^2}\right)\Bigg\{2A\left(-3\,\frac{a''}{a} + 3\left(\frac{a'}{a}\right)^2\right) + (6\psi)\left(\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right) \\
&\quad + -\left(\partial_i\partial^i A + 3\psi'' + 3\frac{a'}{a}\psi' + 3\frac{a'}{a}A'\right) + 3\Bigg[-\frac{a'}{a}A' - 5\frac{a'}{a}\psi' \\
&\quad - 2\frac{a''}{a}A - 2\left(\frac{a'}{a}\right)^2 A - 2\frac{a''}{a}\psi - 2\left(\frac{a'}{a}\right)^2\psi - \psi'' + \partial_k\partial^k\psi\Bigg] \\
&\quad + \left(\partial_i\partial^i\psi - \partial_i\partial^i A\right)\Bigg\} \\
\therefore \delta R &= -\frac{1}{a^2}\Big(2\partial_i\partial^i A + 6\psi'' + 6\frac{a'}{a}A' + 18\frac{a'}{a}\psi' + 12\frac{a''}{a}A - 4\partial_i\partial^i\psi\Big). \qquad \text{(B.40)}
\end{aligned}$$

B.4 Perturbed Einstein Tensor

From

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\,R. \qquad \text{(B.41)}$$

The unperturbed components are

$$\begin{aligned} G_{00} &= R_{00} - \frac{1}{2}\, g_{00}\, R \\ &= -3\,\frac{a''}{a} + 3\left(\frac{a'}{a}\right)^2 + \frac{1}{a^2}\left(6\frac{a''}{a^3}\right) \\ \therefore G_{00} &= 3\Big(\frac{a'}{a}\Big)^2. \end{aligned} \tag{B.42}$$

$$\begin{aligned} G_{0i} &= R_{0i} - \frac{1}{2}\, g_{0i}\, R \\ \therefore G_{0i} &= 0. \end{aligned} \tag{B.43}$$

$$\begin{aligned} G_{ij} &= R_{ij} - \frac{1}{2}\, g_{ij}\, R \\ &= \left(\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right)\delta_{ij} - \frac{1}{2}\big(a^2\delta_{ij}\big)\left(6\frac{a''}{a^3}\right) \\ \therefore G_{ij} &= \left(-2\,\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right)\delta_{ij}. \end{aligned} \tag{B.44}$$

From

$$\delta G_{\mu\nu} = \delta R_{\mu\nu} - \frac{1}{2}\,\delta g_{\mu\nu}\, R - \frac{1}{2}\, g_{\mu\nu}\,\delta R,$$

The perturbed Einstein tensor is

$$\begin{aligned}
\delta G_{00} &= \delta R_{00} - \frac{1}{2}\,\delta g_{00}\,R - \frac{1}{2}\,g_{00}\,\delta R \\
&= \partial_i\partial^i A + 3\psi'' + 3\frac{a'}{a}\psi' + 3\frac{a'}{a}A' - \frac{1}{2}(-2Aa^2)\left(6\frac{a''}{a^3}\right) \\
&\quad - \left(-\frac{1}{2}a^2\right)\left(-\frac{1}{a^2}\right)\Bigg(2\partial_i\partial^i A + 6\psi'' + 6\frac{a'}{a}A' + 18\frac{a'}{a}\psi' \\
&\quad + 12\frac{a''}{a}A - 4\partial_i\partial^i\psi\Bigg) \\
\therefore \delta G_{00} &= -6\frac{a'}{a}\,\psi' + 2\,\partial_i\partial^i\,\psi. && \text{(B.45)} \\
\delta G_{0i} &= \delta R_{0i} - \frac{1}{2}\,\delta g_{0i}\,R - \frac{1}{2}\,g_{0i}\,\delta R \\
&= \delta R_{0i} \\
\therefore \delta G_{0i} &= 2\partial_i\,\psi' + 2\,\frac{a'}{a}\,\partial_i A. && \text{(B.46)} \\
\delta G_{ij} &= \delta R_{ij} - \frac{1}{2}\,\delta g_{ij}\,R - \frac{1}{2}\,g_{ij}\,\delta R, \\
&= \Bigg(-\frac{a'}{a}A' - 5\frac{a'}{a}\psi' - 2\frac{a''}{a}A - 2\left(\frac{a'}{a}\right)^2 A - 2\frac{a''}{a}\psi \\
&\quad - 2\left(\frac{a'}{a}\right)^2\psi - \psi'' + \partial_k\partial^k\psi\Bigg)\delta_{ij} + \partial_i\partial_j\psi - \partial_i\partial_j A \\
&\quad - \frac{1}{2}(-2\psi a^2\delta_{ij})\left(6\frac{a''}{a^3}\right) - \frac{1}{2}(a^2\delta_{ij})\left(\frac{1}{a^2}\right)\left(-\frac{1}{a^2}\right)\Big(2\partial_k\partial^k A \\
&\quad + 6\psi'' + 6\frac{a'}{a}A' + 18\frac{a'}{a}\psi' + 12\frac{a''}{a}A - 4\partial_k\partial^k\psi\Big) \\
\therefore \delta G_{ij} &= \Bigg(2\frac{a'}{a}\,A' + 4\frac{a'}{a}\,\psi' + 4\frac{a''}{a}\,A - 2\left(\frac{a'}{a}\right)^2 A + 4\frac{a''}{a}\,\psi \\
&\quad - 2\left(\frac{a'}{a}\right)^2\psi + 2\psi'' - \partial_k\partial^k\,\psi + \partial_k\partial^k A\Bigg)\delta_{ij} + \partial_i\partial_j\psi \\
&\quad - \partial_i\partial_j A. && \text{(B.47)}
\end{aligned}$$

From

$$\delta G^{\mu}{}_{\nu} = \delta g^{\mu\alpha}\,G_{\alpha\nu} + g^{\mu\alpha}\,\delta G_{\alpha\nu}. \tag{B.48}$$

So

$$
\begin{aligned}
\delta G^0{}_0 &= \delta g^{0\alpha}\, G_{\alpha 0} \,+\, g^{0\alpha}\, \delta G_{\alpha 0} \\
&= \delta g^{00}\, G_{00} \,+\, g^{00}\, \delta G_{00} \\
&= \frac{2A}{a^2}\left(3\left(\frac{a'}{a}\right)^2\right) \,-\, \frac{1}{a^2}\left(6\frac{a'}{a}\psi' \,-\, 2\partial_i\partial^i\psi\right) \\
\therefore\, \delta G^0{}_0 &= \frac{1}{a^2}\left(6\left(\frac{a'}{a}\right)^2 A \,+\, 6\frac{a'}{a}\psi' \,-\, 2\partial_i\partial^i\psi\right). \qquad \text{(B.49)} \\
\delta G^0{}_i &= \delta g^{0\alpha}\, G_{\alpha i} \,+\, g^{0\alpha}\, \delta G_{\alpha i} \\
&= \delta g^{00}\, G_{0i} \,+\, g^{00}\, \delta G_{0i} \\
\therefore\, \delta G^0{}_i &= \frac{1}{a^2}\left(-2\partial_i\psi' \,-\, 2\frac{a'}{a}\partial_i A\right). \qquad \text{(B.50)} \\
\delta G^i{}_j &= \delta g^{i\alpha}\, G_{\alpha j} \,+\, g^{i\alpha}\, \delta G_{\alpha j} \\
&= \delta g^{il}\, G_{lj} \,+\, g^{il}\, \delta G_{lj} \\
&= \frac{2\psi}{a^2}\delta^{il}\left(-2\,\frac{a''}{a} \,+\, \left(\frac{a'}{a}\right)^2\right)\delta_{lj} \,+\, \left(\frac{1}{a^2}\delta^{il}\right)\Bigg\{\Bigg[2\frac{a'}{a}\,A' \\
&\quad + 4\frac{a'}{a}\psi' \,+\, 4\frac{a''}{a}\,A \,-\, 2\left(\frac{a'}{a}\right)^2 A \,+\, 4\frac{a''}{a}\psi \,-\, 2\left(\frac{a'}{a}\right)^2\psi \\
&\quad + 2\psi'' \,-\, \partial_k\partial^k\,\psi \,+\, \partial_k\partial^k A\Bigg]\delta_{lj} + \partial_l\partial_j\psi \,-\, \partial_l\partial_j A\Bigg\} \\
&= \frac{1}{a^2}\Bigg\{-4\psi\frac{a''}{a}\delta^i{}_j \,+\, 2\psi\left(\frac{a'}{a}\right)^2\delta^i{}_j \,+\, \Bigg[2\frac{a'}{a}\,A' \\
&\quad + 4\frac{a'}{a}\psi' \,+\, 4\frac{a''}{a}\,A \,-\, 2\left(\frac{a'}{a}\right)^2 A \,+\, 4\frac{a''}{a}\psi \,-\, 2\left(\frac{a'}{a}\right)^2\psi \\
&\quad + 2\psi'' \,-\, \partial_k\partial^k\,\psi \,+\, \partial_k\partial^k A\Bigg]\delta^i{}_j \,+\, \partial^i\partial_j\psi \,-\, \partial^i\partial_j A\Bigg\} \\
\therefore\, \delta G^i{}_j &= \frac{1}{a^2}\Bigg\{\Bigg(2\,\frac{a'}{a}A' + 4\,\frac{a''}{a}A - 2\left(\frac{a'}{a}\right)^2 A + \partial_k\partial^k A \,+\, 4\,\frac{a'}{a}\psi' \\
&\quad + 2\,\psi'' - \partial_k\partial^k\psi\Bigg)\delta^i{}_j - \partial^i\partial_j A + \partial^i\partial_j\psi\Bigg\}. \qquad \text{(B.51)}
\end{aligned}
$$

B.5 Perturbed Energy-Momentum Tensor

From

$$
T_{\mu\nu} \,=\, \partial_\mu\phi\,\partial_\nu\phi \,-\, g_{\mu\nu}\left(\frac{1}{2}\,g^{\alpha\beta}\,\partial_\alpha\phi\,\partial_\beta\phi \,+\, V(\phi)\right).
$$

The background components are

$$
\begin{aligned}
T_{00} &= \partial_0\phi\,\partial_0\phi \,-\, g_{00}\left(\frac{1}{2}\,g^{\alpha\beta}\,\partial_\alpha\phi\,\partial_\beta\phi \,+\, V(\phi)\right) \\
&= \phi'^2 - (-a^2)\left(-\frac{1}{2a^2}\phi'^2 \,+\, \frac{1}{2a^2}\delta^{ij}\partial_i\phi\partial_j\phi \,+\, V(\phi)\right) \\
&= \frac{1}{2}\,\phi'^2 \,+\, \frac{(\nabla\phi)^2}{2} \,+\, V(\phi)\,a^2 \\
\therefore T_{00} &= \frac{1}{2}\,\phi'^2 \,+\, V(\phi)\,a^2. \qquad \text{(B.52)} \\
T_{0i} &= \partial_0\phi\,\partial_i\phi \,-\, g_{0i}\left(\frac{1}{2}\,g^{\alpha\beta}\,\partial_\alpha\phi\,\partial_\beta\phi \,+\, V(\phi)\right) \\
&= \partial_0\phi\partial_i\phi \quad = \quad \phi'\nabla\phi \\
\therefore T_{0i} &= 0. \qquad \text{(B.53)} \\
T_{ij} &= \partial_i\phi\,\partial_j\phi \,-\, g_{ij}\left(\frac{1}{2}\,g^{\alpha\beta}\,\partial_\alpha\phi\,\partial_\beta\phi \,+\, V(\phi)\right) \\
&= \partial_i\phi\,\partial_j\phi \,-\, (a^2\delta_{ij})\left(-\frac{1}{2a^2}\phi'^2 \,+\, \frac{1}{2a^2}\delta^{ij}\partial_i\phi\partial_j\phi \,+\, V(\phi)\right) \\
&= \left(\frac{1}{2}\,\phi'^2 \,-\, \frac{(\nabla\phi)^2}{2} \,-\, V(\phi)\,a^2\right)\,\delta_{ij} \\
\therefore T_{ij} &= \left(\frac{1}{2}\,\phi'^2 \,-\, V(\phi)\,a^2\right)\,\delta_{ij}, \qquad \text{(B.54)}
\end{aligned}
$$

while $\partial_i\phi \,=\, \nabla\phi \,=\, 0$ because the background field ϕ is homogeneous.
From

$$
\begin{aligned}
\delta T_{\mu\nu} \;=\; & \partial_\mu\delta\phi\,\partial_\nu\phi \,+\, \partial_\mu\phi\,\partial_\nu\delta\phi \,-\, \delta g_{\mu\nu}\left(\frac{1}{2}\,g^{\alpha\beta}\,\partial_\alpha\phi\,\partial_\beta\phi \,+\, V(\phi)\right) \\
& -\, g_{\mu\nu}\left(\frac{1}{2}\delta g^{\alpha\beta}\,\partial_\alpha\phi\,\partial_\beta\phi + g^{\alpha\beta}\,\partial_\alpha\delta\phi\,\partial_\beta\phi + V_\phi\,\delta\phi\right).
\end{aligned}
$$

The perturbed components are

$$\begin{aligned}
\delta T_{00} &= \partial_0\delta\phi\,\partial_0\phi \,+\, \partial_0\phi\,\partial_0\delta\phi \,-\, \delta g_{00}\left(\frac{1}{2}\,g^{\alpha\beta}\,\partial_\alpha\phi\,\partial_\beta\phi \,+\, V(\phi)\right)\\
&\quad - g_{00}\left(\frac{1}{2}\delta g^{\alpha\beta}\,\partial_\alpha\phi\,\partial_\beta\phi + g^{\alpha\beta}\,\partial_\alpha\delta\phi\,\partial_\beta\phi + V_\phi\,\delta\phi\right)\\
&= \partial_0\delta\phi\,\partial_0\phi \,+\, \partial_0\phi\,\partial_0\delta\phi \,-\, \delta g_{00}\left(\frac{1}{2}\,g^{00}\,\partial_0\phi\,\partial_0\phi \,+\, \frac{1}{2}\,g^{ij}\,\partial_i\phi\,\partial_j\phi \,+\, V(\phi)\right)\\
&\quad - g_{00}\Bigg(\frac{1}{2}\delta g^{00}\,\partial_0\phi\,\partial_0\phi \,+\, \frac{1}{2}\delta g^{ij}\,\partial_i\phi\,\partial_j\phi \,+\, g^{00}\,\partial_0\delta\phi\,\partial_0\phi\\
&\quad + g^{ij}\,\partial_i\delta\phi\,\partial_j\phi \,+\, V_\phi\,\delta\phi\Bigg)\\
&= 2\phi'\delta\phi \,-\, (-2Aa^2)\left(\frac{-1}{2a^2}\phi'^2 \,+\, \frac{1}{2a^2}\delta^{ij}\partial_i\phi\,\partial_j\phi + V(\phi)\right)\\
&\quad - (-a^2)\Bigg[\frac{1}{2}\left(\frac{2A}{a^2}\right)\phi'^2 \,+\, \frac{1}{2}\left(\frac{2\psi}{a^2}\delta^{ij}\right)\partial_i\phi\,\partial_j\phi \,+\, \left(-\frac{1}{a^2}\right)\phi'\delta\phi'\\
&\quad + \frac{1}{a^2}\left(\delta^{ij}\right)\partial_i\delta\phi\,\partial_j\phi \,+\, V_\phi\,\delta\phi\Bigg]\\
\therefore\,\delta T_{00} &= \delta\phi'\,\phi' \,+\, 2\,A\,V(\phi)\,a^2 \,+\, a^2\,V_\phi\,\delta\phi. \qquad\qquad (B.55)\\
\delta T_{0i} &= \partial_0\delta\phi\,\partial_i\phi \,+\, \partial_0\phi\,\partial_i\delta\phi \,-\, \delta g_{0i}\left(\frac{1}{2}\,g^{\alpha\beta}\,\partial_\alpha\phi\,\partial_\beta\phi \,+\, V(\phi)\right)\\
&\quad - g_{0i}\left(\frac{1}{2}\delta g^{\alpha\beta}\,\partial_\alpha\phi\,\partial_\beta\phi + g^{\alpha\beta}\,\partial_\alpha\delta\phi\,\partial_\beta\phi + V_\phi\,\delta\phi\right)\\
&= 0 \,+\, \phi'\partial_i\delta\phi \,-\, 0 \,-\, 0\\
\therefore\,\delta T_{0i} &= \partial_i\,\delta\phi\,\phi'. \qquad\qquad (B.56)\\
\delta T_{ij} &= \partial_i\delta\phi\,\partial_j\phi \,+\, \partial_i\phi\,\partial_j\delta\phi \,-\, \delta g_{ij}\left(\frac{1}{2}\,g^{\alpha\beta}\,\partial_\alpha\phi\,\partial_\beta\phi \,+\, V(\phi)\right)\\
&\quad - g_{ij}\left(\frac{1}{2}\delta g^{\alpha\beta}\,\partial_\alpha\phi\,\partial_\beta\phi + g^{\alpha\beta}\,\partial_\alpha\delta\phi\,\partial_\beta\phi + V_\phi\,\delta\phi\right)\\
&= 0 \,+\, 0 \,-\, \delta g_{ij}\left(\frac{1}{2}\,g^{00}\,\partial_0\phi\,\partial_0\phi \,+\, \frac{1}{2}\,g^{lk}\,\partial_l\phi\,\partial_k\phi \,+\, V(\phi)\right)\\
&\quad - g_{ij}\Bigg(\frac{1}{2}\delta g^{00}\,\partial_0\phi\,\partial_0\phi \,+\, \frac{1}{2}\delta g^{lk}\,\partial_l\phi\,\partial_k\phi \,+\, g^{00}\,\partial_0\delta\phi\,\partial_0\phi\\
&\quad + g^{lk}\,\partial_l\delta\phi\,\partial_k\phi \,+\, V_\phi\,\delta\phi\Bigg)\\
&= -\,(2\psi a^2\delta_{ij})\left(-\frac{1}{2a^2}\phi'^2 \,+\, V(\phi)\right) \,-\, a^2\delta_{ij}\left(\frac{A}{a^2}\phi'^2 \,-\, \frac{\phi'\delta\phi'}{a^2} + V_\phi\delta\phi\right)\\
\therefore\,\delta T_{ij} &= \left(\delta\phi'\,\phi' - A\,\phi'^2 - a^2\,V_\phi\,\delta\phi - \psi\,\phi'^2 + 2\,\psi\,V(\phi)\,a^2\right)\delta_{ij}\,. \qquad\qquad (B.57)
\end{aligned}$$

From

$$\begin{aligned}\delta T^{\mu}{}_{\nu} &= \delta(g^{\mu\alpha}\,T_{\alpha\nu}) \\ &= \delta g^{\mu\alpha}\,T_{\alpha\nu} \,+\, g^{\mu\alpha}\,\delta T_{\alpha\nu},\end{aligned}$$

so

$$\begin{aligned}\delta T^{0}{}_{0} &= \delta g^{0\alpha}\,T_{\alpha 0} \,+\, g^{0\alpha}\,\delta T_{\alpha 0} \\ &= \delta g^{00}\,T_{00} \,+\, g^{00}\,\delta T_{00} \\ &= \frac{2A}{a^2}\left(\frac{1}{2}\,\phi'^2 \,+\, V(\phi)\,a^2\right) - \frac{1}{a^2}\left(\delta\phi'\,\phi' \,+\, 2\,A\,V(\phi)\,a^2 \,+\, a^2\,V_\phi\,\delta\phi\right) \\ \therefore\, \delta T^{0}{}_{0} &= \frac{1}{a^2}\left(A\,\phi'^2 \,-\, \delta\phi'\,\phi' \,-\, \delta\phi\,\frac{\partial V}{\partial\phi}\,a^2\right). \qquad \text{(B.58)} \\ \delta T^{0}{}_{i} &= \delta g^{0\alpha}\,T_{\alpha i} \,+\, g^{0\alpha}\,\delta T_{\alpha i} \\ &= \delta g^{00}\,T_{0i} \,+\, g^{00}\,\delta T_{0i} \\ \therefore\, \delta T^{0}{}_{i} &= \frac{1}{a^2}\left(-\,\partial_i\delta\phi\,\phi'\right). \qquad \text{(B.59)} \\ \delta T^{i}{}_{j} &= \delta g^{i\alpha}\,T_{\alpha j} \,+\, g^{i\alpha}\,\delta T_{\alpha j} \\ &= \delta g^{il}\,T_{lj} \,+\, g^{il}\,\delta T_{lj} \\ &= \frac{2\psi}{a^2}\delta^{il}\left(\frac{1}{2}\,\phi'^2 \,-\, V(\phi)\,a^2\right)\delta_{lj} \\ &\quad + \frac{1}{a^2}\delta_{lj}\left(\delta\phi'\,\phi' - A\,\phi'^2 - a^2\,V_\phi\,\delta\phi - \psi\,\phi'^2 + 2\,\psi\,V(\phi)\,a^2\right)\delta_{lj} \\ \therefore\, \delta T^{i}{}_{j} &= \frac{1}{a^2}\left(-\,A\,\phi'^2 \,+\, \delta\phi'\,\phi' \,-\, \delta\phi\,\frac{\partial V}{\partial\phi}\,a^2\right)\delta^{i}{}_{j}. \qquad \text{(B.60)}\end{aligned}$$

B.6 Perturbed Einstein Equation

The 00-unperturbed component is

$$\begin{aligned}G_{00} &= \frac{1}{m_{pl}^2}T_{00} \\ 3\left(\frac{a'}{a}\right)^2 &= \frac{1}{m_{pl}^2}\left(\frac{1}{2}\,\phi'^2 \,+\, V(\phi)\,a^2\right) \\ \left(\frac{a'}{a}\right)^2 &= \frac{1}{3m_{pl}^2}\left(\frac{1}{2}\,\phi'^2 \,+\, V(\phi)\,a^2\right). \qquad \text{(B.61)}\end{aligned}$$

The ij-unperturbed component is

$$\begin{aligned}
G_{ij} &= \frac{1}{m_{pl}^2} T_{ij} \\
\left(-2\,\frac{a''}{a} + \left(\frac{a'}{a}\right)^2 \right) \delta_{ij} &= \frac{1}{m_{pl}^2} \left(\frac{1}{2}\,\phi'^2 - V(\phi)\,a^2 \right) \delta_{ij} \\
2\,\frac{a''}{a} - \left(\frac{a'}{a}\right)^2 &= \frac{1}{m_{pl}^2} \left(V(\phi)\,a^2 - \frac{1}{2}\,\phi'^2 \right) \\
2\,\frac{a''}{a} - 2\left(\frac{a'}{a}\right)^2 &= -\left(\frac{a'}{a}\right)^2 + \frac{1}{m_{pl}^2} \left(V(\phi)\,a^2 - \frac{1}{2}\,\phi'^2 \right). \quad \text{(B.62)}
\end{aligned}$$

Substitute (B.61) in the right hand side, one obtains

$$\frac{a''}{a} - \left(\frac{a'}{a}\right)^2 = \frac{1}{3m_{pl}^2} \left(V(\phi)\,a^2 - \phi'^2 \right). \quad \text{(B.63)}$$

From

$$\delta G^{\mu}{}_{\nu} = \frac{1}{m_{pl}^2} \delta T^{\mu}{}_{\nu}, \quad \text{(B.64)}$$

one obtains

$$\begin{aligned}
\delta G^{0}{}_{0} &= \frac{1}{m_{pl}^2} \delta T^{0}{}_{0} \\
\frac{1}{a^2} \left(6\mathcal{H}^2 A + 6\mathcal{H}\psi' - 2\partial_i \partial^i \psi \right) &= \frac{1}{m_{pl}^2} \left(\frac{1}{a^2} \right) \left(A\,\phi'^2 - \delta\phi'\,\phi' - \delta\phi\,\frac{\partial V}{\partial \phi}\,a^2 \right) \\
3\mathcal{H}^2 \psi + 3\mathcal{H}\psi' - \nabla^2 \psi &= \frac{1}{2m_{pl}^2} \left(\psi\phi'^2 - \delta\phi'\phi' - \delta\phi V_{\phi} a^2 \right). \quad \text{(B.65)} \\
\delta G^{0}{}_{i} &= \frac{1}{m_{pl}^2} \delta T^{0}{}_{i} \\
\frac{1}{a^2} \left(-2\partial_i \psi' - 2\mathcal{H}\partial_i A \right) &= \frac{1}{m_{pl}^2} \left(\frac{1}{a^2} \right) \left(-\,\partial_i \delta\phi\,\phi' \right) \\
\partial_i \psi' + \mathcal{H}\partial_i \psi &= \frac{1}{2m_{pl}^2} \left(\partial_i(\delta\phi\,\phi') - (\partial_i \phi')\delta\phi \right) \\
\partial_i(\mathcal{H}\psi + \psi') &= \frac{1}{2m_{pl}^2} \partial_i(\delta\phi\,\phi') \\
\mathcal{H}\psi + \psi' &= \frac{1}{2m_{pl}^2} \left(\delta\phi\phi' \right). \quad \text{(B.66)}
\end{aligned}$$

$$\begin{aligned}
\delta G^i{}_j &= \frac{1}{m_{pl}^2}\delta T^i{}_j \\
\frac{1}{a^2}\Bigg(2\,\mathcal{H}A' + 4\,\left(\mathcal{H}^2 + \mathcal{H}'\right)A &- 2\left(\mathcal{H}\right)^2 A + \partial_k\partial^k A + 4\,\mathcal{H}\psi' \\
+\,2\,\psi'' - \partial_k\partial^k\psi\Bigg)\delta^i{}_j = \frac{1}{m_{pl}^2}\left(\frac{1}{a^2}\right)\Bigg(&- A\,\phi'^2 + \delta\phi'\,\phi' - \delta\phi\,\frac{\partial V}{\partial\phi}\,a^2\Bigg)\delta^i{}_j \\
\mathcal{H}^2\psi + 2\mathcal{H}'\psi + 3\mathcal{H}\psi' + \psi'' &= \frac{1}{2m_{pl}^2}\left(-\psi\phi'^2 + \delta\phi'\phi' - \delta\phi V_\phi a^2\right) \quad \text{(B.67)}
\end{aligned}$$

APPENDIX C

POWER SPECTRUM AND THE ACTION

C.1 Calculation of $\frac{z''}{z}$

From $z = \frac{a\phi'}{\mathcal{H}} = \frac{a\dot{\phi}}{H}$,

$$\frac{dz}{d\eta} \;=\; a\frac{dz}{dt} \quad = a\left[\frac{\dot{a}\dot{\phi}}{H} + \frac{\dot{a}\ddot{\phi}}{H} - \frac{a\dot{\phi}\dot{H}}{H^2}\right].$$

where $\frac{\dot{a}}{H} = a$. Next one computes

$$\begin{aligned}\frac{d^2z}{d\eta^2} &= a\frac{d}{dt}\left(a\frac{dz}{dt}\right)\\ &= a\Bigg\{\dot{a}\left[a\dot{\phi} + \frac{\dot{a}\ddot{\phi}}{H} - \frac{a\dot{\phi}\dot{H}}{H^2}\right] + a\Bigg[\dot{a}\dot{\phi} + a\ddot{\phi} + \frac{\dot{a}\ddot{\phi}}{H} + \frac{a\dddot{\phi}}{H}\\ &\quad - 2\frac{a\ddot{\phi}\dot{H}}{H^2} - \frac{\dot{a}\dot{\phi}\dot{H}}{H^2} - \frac{a\dot{\phi}\ddot{H}}{H^2} + 2\frac{a\dot{\phi}\dot{H}^2}{H^3}\Bigg]\Bigg\}\end{aligned}$$

$$
\begin{aligned}
\frac{1}{z}\frac{d^2z}{d\eta^2} &= \frac{H}{\dot\phi}\Bigg\{\dot a\Bigg[a\dot\phi + \frac{\dot a\ddot\phi}{H} - \frac{a\dot\phi\dot H}{H^2}\Bigg] + a\Bigg[\dot a\dot\phi + 2a\ddot\phi + \frac{a\dddot\phi}{H} \\
&\quad - 2\frac{a\ddot\phi\dot H}{H^2} - \frac{a\dot\phi\dot H}{H} - \frac{a\dot\phi\ddot H}{H^2} + 2\frac{a\dot\phi\dot H^2}{H^3}\Bigg]\Bigg\} \\
&= \frac{(aH)^2}{\dot\phi}\Bigg\{\frac{\dot a}{a^2H}\Bigg[a\dot\phi + a\dot\phi\delta_1 + a\dot\phi\epsilon\Bigg] + \frac{1}{aH}\Bigg[\dot a\dot\phi + 2a\ddot\phi + \frac{a\dddot\phi}{H} \\
&\quad + 2a\ddot\phi\epsilon - \frac{a\dot\phi\dot H}{H} + a\dot\phi\Bigg(-\frac{\ddot H}{H^2} + 2\frac{\dot H^2}{H^3}\Bigg)\Bigg]\Bigg\} \\
&= (aH)^2\Bigg\{\Bigg[1 + \delta_1 + \epsilon\Bigg] + \Bigg[1 + 2\delta_1 + \delta_2 \\
&\quad + 2\epsilon\delta_1 + \epsilon + \frac{\dot\epsilon}{H}\Bigg]\Bigg\} \\
\therefore \frac{z''}{z} &= 2(aH)^2\Bigg(1 + \epsilon + \frac{3}{2}\delta_1 + \frac{\delta_2}{2} + \epsilon^2 + 2\epsilon\delta_1\Bigg). \qquad \text{(C.1)}
\end{aligned}
$$

where $\epsilon = -\frac{\dot H}{H^2}$, $\delta_1 = \frac{\ddot\phi}{\dot\phi H}$, $\delta_2 = \frac{\dddot\phi}{\dot\phi H^2}$ and [see appendix C.4] $\dot\epsilon = 2H(\epsilon^2 + \epsilon\delta_1)$.

C.2 Commutation Relation Gives Norm of the Field $u(\eta, \vec{x})$

From the commutator

$$[\varphi(\eta, \vec{x}), \pi(\eta', \vec{x}')]_{\eta=\eta'} = i\delta^{(3)}(\vec{x} - \vec{x}'),$$

for $\vec{x} = \vec{x}'$, one has

$$[\varphi(\eta, \vec{x}), \pi(\eta', \vec{x})]_{\eta=\eta'} = i. \qquad \text{(C.2)}$$

From the lagrangian of the field, the canonical momentum is

$$
\begin{aligned}
\pi &= \frac{\partial\mathcal{L}}{\partial\dot\varphi} \\
&= a^2\varphi'. \qquad \text{(C.3)}
\end{aligned}
$$

The commutator becomes

$$
\begin{aligned}
[\varphi\, ,\, a^2\varphi'] &= i \\
a^2(\varphi_k{\varphi'_k}^* - \varphi_k^*\varphi'_k) &= i.
\end{aligned}
$$

Consider $\upsilon \equiv a\varphi$, one obtains

$$\upsilon_k \upsilon_k'^{\,*} - \upsilon_k^* \upsilon_k' \;=\; i. \tag{C.4}$$

The relation between υ_k and υ_k' which satisfies the condition above is

$$\upsilon_k' \;=\; -i\omega_k \upsilon_k,$$

where ω_k is the positive frequency at mode k. Now we can compute the norm of υ_k

$$\begin{aligned} \upsilon_k(i\omega_k \upsilon_k^*) - \upsilon_k^*(-i\omega_k \upsilon_k) &= i \\ \upsilon_k^* \upsilon_k &= \frac{1}{2\omega_k} \\ \therefore |\upsilon_k| &= \frac{1}{\sqrt{2\omega_k}}. \end{aligned} \tag{C.5}$$

From $u \equiv a\delta\phi + z\psi = \upsilon + z\psi$, its norm can be computed as follow (using (C.4))

$$u_k u_k'^{\,*} - u_k^* u_k' \;=\; i, \tag{C.6}$$

so

$$|u_k| \;=\; |\upsilon_k| \;=\; \frac{1}{\sqrt{2\omega_k}}. \tag{C.7}$$

For the equation of motion in small scale limit $u_k'' + k^2 u_k \approx 0$, the positive frequency is $\omega_k = k$. Therefore

$$|u_k| = \frac{1}{\sqrt{2k}}. \tag{C.8}$$

C.3 Vacuum State and Power Spectrum

Consider vacuum state

$$\begin{aligned} \langle 0|u^*(\eta,\vec{x})u(\eta',\vec{x}')|0\rangle_{\eta=\eta'} &= \langle 0| \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \left(u_k^*(\eta)\hat{a}_{\vec{k}}^\dagger e^{-i\vec{k}\cdot\vec{x}} + u_k(\eta)\hat{a}_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} \right) \\ &\quad \left(u_p(\eta)\hat{a}_{\vec{p}} e^{i\vec{p}\cdot\vec{x}'} + u_p^*(\eta)\hat{a}_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}'} \right) |0\rangle, \end{aligned}$$

Using $a_{\vec{k}}|0\rangle = 0$ and $[\hat{a}_{\vec{k}}, \hat{a}_{\vec{p}}^\dagger] = \delta^{(3)}(\vec{k}-\vec{p})$

$$\begin{aligned} \langle 0|u^*(\eta,\vec{x})u(\eta',\vec{x}')|0\rangle_{\eta=\eta'} &= \int \frac{d^3k d^3p}{(2\pi)^3} u_k(\eta)u_p^*(\eta) \left(e^{i(\vec{k}\cdot\vec{x}-\vec{p}\cdot\vec{x}')} \right) \langle 0|\hat{a}_{\vec{k}}\hat{a}_{\vec{p}}^\dagger|0\rangle \\ &= \int \frac{d^3k d^3p}{(2\pi)^3} u_k u_p^* \left(e^{i(\vec{k}\cdot\vec{x}-\vec{p}\cdot\vec{x}')} \right) \langle 0| \left([\hat{a}_{\vec{k}}, \hat{a}_{\vec{p}}^\dagger] + \hat{a}_{\vec{k}}^\dagger \hat{a}_{\vec{p}} \right) |0\rangle \\ &= \int \frac{d^3k d^3p}{(2\pi)^3} u_k u_p^* \left(e^{i(\vec{k}\cdot\vec{x}-\vec{p}\cdot\vec{x}')} \right) \delta^{(3)}(\vec{k}-\vec{p}) \\ \therefore \langle u^*(\eta,\vec{x})u(\eta,\vec{x}')\rangle &= \int \frac{d^3k}{(2\pi)^3} |u_k(\eta)|^2 e^{i\vec{k}(\vec{x}-\vec{x}')}, \end{aligned} \tag{C.9}$$

so

$$\begin{aligned}\langle u^2\rangle &= \int \frac{d^3k}{(2\pi)^3}|u_k(\eta)|^2 &&= 4\pi\int\frac{k^2dk}{(2\pi)^3}|u_k(\eta)|^2\\ &= \int\frac{dk}{k}\left(\frac{k^3}{2\pi^2}|u_k(\eta)|^2\right) &&\equiv \int\frac{dk}{k}\mathcal{P}_u(k),\end{aligned}\tag{C.10}$$

where

$$\mathcal{P}_u(k) = \frac{k^3}{2\pi^2}|u_k(\eta)|^2$$

is the *power spectrum* of the perturbation $u(\eta)$ at the scale k. The power spectrum of the curvature perturbation is

$$\begin{aligned}\mathcal{P}_{\mathcal{R}}(k) &= \frac{k^3}{2\pi^2}|\mathcal{R}_k(\eta)|^2 = \frac{k^3}{2\pi^2}\left|\frac{u_k(\eta)}{z}\right|^2\\ &= \frac{k^2}{(2\pi)^2}\frac{2^{2\nu-3}}{z^2}\left[\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right]^2(-k\eta)^{1-2\nu}\\ &= \frac{k^2}{(2\pi)^2}\frac{2^{2\nu-3}}{(a\dot{\phi}/H)^2}\left[\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right]^2\left[\frac{k}{aH(1-\epsilon)}\right]^{1-2\nu}\\ &= \frac{H^4}{(2\pi\dot{\phi})^2}2^{2\nu-3}\left[\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right]^2\left[\frac{k}{aH}\right]^{-4\epsilon-2\delta_1}(1-\epsilon)^{2+4\epsilon+2\delta_1}\\ \therefore \mathcal{P}_{\mathcal{R}}(k) &= -\frac{1}{2m_{pl}^2\epsilon}\left(\frac{H}{2\pi}\right)^2 2^{2\nu-3}\left[\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right]^2\left[\frac{k}{aH}\right]^{-4\epsilon-2\delta_1}(1-\epsilon)^{2+4\epsilon+2\delta_1}\end{aligned}\tag{C.11}$$

where $\eta = \frac{-1}{aH(1-\epsilon)}$ and $z = \frac{a\dot{\phi}}{H}$.
Using Taylor expansion up to the first order,

$$\begin{aligned}2^{\nu-\frac{3}{2}}\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})} &\approx 1+2\alpha(2\epsilon+\delta_1),\\ (1-\epsilon)^{2+4\epsilon+2\delta_1} &\approx (1-2\epsilon-4\epsilon^2-2\epsilon\delta_1),\\ \left(\frac{k}{aH}\right)^{-4\epsilon-2\delta_1} &= e^{(-4\epsilon-2\delta_1)\ln\left(\frac{k}{aH}\right)} \approx 1-(4\epsilon+2\delta_1)\ln\left(\frac{k}{aH}\right),\end{aligned}\tag{C.12}$$

so

$$2^{2\nu-3}\left[\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right]^2 \approx 1+8\alpha\epsilon+4\alpha\delta_1,\tag{C.13}$$

The power spectrum becomes

$$\begin{aligned}\mathcal{P}_{\mathcal{R}}(k) &= -\frac{1}{2m_{pl}^2\epsilon}\left(\frac{H}{2\pi}\right)^2\Bigg[\left(1-(4\epsilon+2\delta_1)\ln\left(\frac{k}{aH}\right)\right)(1+8\alpha\epsilon+4\alpha\delta_1)\\ &\qquad \left(1-2\epsilon-4\epsilon^2-2\epsilon\delta_1\right)\Bigg]\\ &= -\frac{1}{2m_{pl}^2\epsilon}\left(\frac{H}{2\pi}\right)^2\left[1-2\epsilon+2\left(\alpha-\ln\left(\frac{k}{aH}\right)\right)(2\epsilon+\delta_1)\right].\end{aligned}\tag{C.14}$$

C.4 Spectral Index and Running

From the definition, the spectral index is

$$n_{\mathcal{R}}(k) = 1 + \frac{d\ln \mathcal{P}_{\mathcal{R}}}{d\ln k} \quad = \quad 1 + \frac{1}{\mathcal{P}_{\mathcal{R}}}\frac{d\mathcal{P}_{\mathcal{R}}}{d\ln k}, \tag{C.15}$$

where the power spectrum is

$$\mathcal{P}_{\mathcal{R}}(k) \quad = \quad \frac{1}{2m_{pl}^2\epsilon}\left(\frac{H}{2\pi}\right)^2\left[1 - 2\epsilon + 2\left(\alpha - \ln\left(\frac{k}{aH}\right)\right)(2\epsilon + \delta_1)\right]. \tag{C.16}$$

Computing $\dot{\epsilon}$, $\dot{\delta}_1$, $\dot{\delta}_2$ and $\dot{\mu}$ ($\dot{\mu}$ is computed for Chapter IV).

$$\begin{aligned}
\epsilon &= -\frac{\dot{\phi}^2}{2m_{pl}^2H^2} \\
\dot{\epsilon} &= -\frac{1}{2m_{pl}^2}\left[\frac{2\dot{\phi}\ddot{\phi}}{H^2} - \frac{2\dot{\phi}^2\dot{H}}{H^3}\right] \quad = \quad 2H\epsilon\delta_1 + 2H\epsilon^2 \\
\therefore \dot{\epsilon} &= 2H(\epsilon^2 + \epsilon\delta_1), && \text{(C.17)} \\
\delta_1 &= \frac{\ddot{\phi}}{\dot{\phi}H} \\
\dot{\delta}_1 &= \frac{\dddot{\phi}}{\dot{\phi}H} - \frac{\ddot{\phi}^2}{\dot{\phi}^2H} - \frac{\ddot{\phi}\dot{H}}{\dot{\phi}H^2} \quad = \quad H\delta_2 - H\delta_1^2 + H\epsilon\delta_1 \\
\therefore \dot{\delta}_1 &= H(\epsilon\delta_1 - \delta_1^2 + \delta_2), && \text{(C.18)} \\
\delta_2 &= \frac{\dddot{\phi}}{\dot{\phi}H^2} \\
\dot{\delta}_2 &= \frac{\ddddot{\phi}}{\dot{\phi}H^2} - \frac{\dddot{\phi}\ddot{\phi}}{\dot{\phi}^2H^2} - \frac{2\dddot{\phi}\dot{H}}{\dot{\phi}H^3} \quad = \quad H\delta_3 - H\delta_1\delta_2 + 2H\epsilon\delta_2 \\
\therefore \dot{\delta}_2 &= H(2\epsilon\delta_2 - \delta_1\delta_2 + \delta_3), && \text{(C.19)} \\
\mu &= \left(\frac{kH}{aM_s^2}\right)^2 \quad = \quad 2\left(\frac{H}{M_s}\right)^4, \quad (k \approx \sqrt{2}aH) \\
\dot{\mu} &= 4\left(\frac{H}{M_s}\right)^3\frac{\dot{H}}{M_s} \quad = \quad 4\left[2\left(\frac{H}{M_s}\right)^4\right]\frac{\dot{H}}{H} \quad = \quad -4H\mu\epsilon \\
\therefore \dot{\mu} &= -4H\mu\epsilon. && \text{(C.20)}
\end{aligned}$$

In addition

$$\frac{dH^2}{d\ln k} \quad = \quad \frac{2\dot{H}}{1-\epsilon} \quad \approx -2\dot{H}\epsilon, \tag{C.21}$$

$$\frac{d\epsilon^{-1}}{d\ln k} \quad = \quad -2 - 2\frac{\delta_1}{\epsilon} \tag{C.22}$$

In the crossing scale, one computes

$$\begin{aligned}
\frac{\mathcal{P}_{\mathcal{R}}}{d\ln k} &= \left(\frac{H}{2\pi m_{pl}}\right)^2\left[\left(-2 - 2\frac{\delta_1}{\epsilon}\right)(1 - 2\epsilon + 2\alpha(2\epsilon + \delta_1))\right] \\
&= \frac{1}{\epsilon}[-2\epsilon^2 - 2\epsilon\delta_1 + \alpha(4\epsilon^2 + 5\epsilon\delta_1 - \delta_1^2 - \delta_2)].
\end{aligned}$$

$$
\begin{aligned}
n_{\mathcal{R}} - 1 &= 2\left[(-2\epsilon - \delta_1) + \frac{(-2\epsilon^2 - 2\epsilon\delta_1 + \alpha(4\epsilon^2 + 5\epsilon\delta_1 - \delta_1^2 - \delta_2))}{1 - 2\epsilon + 2\alpha(2\epsilon + \delta_1)}\right] \\
&\approx 2\left[(-2\epsilon - \delta_1) + (-2\epsilon^2 - 2\epsilon\delta_1 + \alpha(4\epsilon^2 + 5\epsilon\delta_1 - \delta_1^2 - \delta_2))\right] \quad \text{(C.23)}
\end{aligned}
$$

Thus the spectral index is

$$
n_{\mathcal{R}}(k) = 1 - 4\epsilon - 2\delta_1 + (8\alpha - 4)\epsilon^2 + (10\alpha - 4)\epsilon\delta_1 - 2\alpha\delta_1^2 + 2\alpha\delta_2. \quad \text{(C.24)}
$$

In the same procedure, one obtains the running of the spectral index

$$
\begin{aligned}
\frac{dn_{\mathcal{R}}}{d\ln k}(k) &= -8\epsilon^2 - 10\epsilon\delta_1 + 2\delta_1^2 - 2\delta_2 + (32\alpha - 16)\epsilon^3 + (62\alpha - 28)\epsilon^2\delta_1 \\
&\quad + (6\alpha - 4)\epsilon\delta_1^2 + (14\alpha - 4)\epsilon\delta_2 + 4\alpha\delta_1^3 - 6\alpha\delta_1\delta_2 + 2\alpha\delta_3. \quad \text{(C.25)}
\end{aligned}
$$

C.5 Modified Action (Time-Time Component) in the Noncommutative Spacetime

From

$$
\tilde{S} = \int d\tau dx \frac{1}{2}\Big(\partial_\tau\tilde{\phi}^\dagger * a^2 * \partial_\tau\tilde{\phi} - \big(\partial_x\tilde{\phi}\big)^\dagger * a^{-2} * \partial_x\tilde{\phi}\Big). \quad \text{(C.26)}
$$

The time-time component action in momentum space is found to be

$$
\begin{aligned}
\tilde{S}_{time} &= V\int d\tau dx \frac{dkdq}{2\pi}\frac{1}{2}\left[\partial_\tau\Big(\frac{\tilde{\phi}_q^\dagger e^{-iqx} + \tilde{\phi}_q e^{iqx}}{2}\Big) * a^2 * \partial_\tau\Big(\frac{\tilde{\phi}_k e^{ikx} + \tilde{\phi}_k^\dagger e^{-ikx}}{2}\Big)\right] \\
&= V\int d\tau dx \frac{dkdq}{2\pi}\frac{1}{8}\Bigg[\partial_\tau\tilde{\phi}_q^\dagger e^{-iqx} * a^2 * \partial_\tau\tilde{\phi}_k e^{ikx} + \partial_\tau\tilde{\phi}_q e^{iqx} * a^2 * \partial_\tau\tilde{\phi}_k e^{ikx} \\
&\quad \partial_\tau\tilde{\phi}_q^\dagger e^{-iqx} * a^2 * \partial_\tau\tilde{\phi}_k^\dagger e^{-ikx} + \partial_\tau\tilde{\phi}_q e^{iqx} * a^2 * \partial_\tau\tilde{\phi}_k^\dagger e^{-ikx}\Bigg].
\end{aligned}
$$

Using the $*$-product

$$
\begin{aligned}
a^2 * \partial_\tau\tilde{\phi}_k e^{ikx} &= a^2 \cdot e^{-il_s^2(\partial_\tau\partial_x - \partial_x\partial_\tau)}\partial_\tau\tilde{\phi}_k\, e^{ikx} \\
&= \partial_\tau\tilde{\phi}_k\left(e^{k\,l_s^2\,\partial_\tau}a^2\right)e^{ikx}, \\
a^2 * \partial_\tau\tilde{\phi}_k^\dagger e^{-ikx} &= a^2 \cdot e^{-il_s^2(\partial_\tau\partial_x - \partial_x\partial_\tau)}\partial_\tau\tilde{\phi}_k^\dagger\, e^{-ikx} \\
&= \partial_\tau\tilde{\phi}_k^\dagger\left(e^{-k\,l_s^2\,\partial_\tau}a^2\right)e^{-ikx},
\end{aligned}
$$

and $\int(f * g)(\tau, x)dx = \int(f \cdot g)(\tau, x)dx$ due to the antisymmetric property in SSUR, the action is then

$$
\begin{aligned}
\tilde{S}_{time} &= V\int d\tau dx \frac{dkdq}{2\pi}\frac{1}{8}\Bigg[\partial_\tau\tilde{\phi}_q^\dagger\,\partial_\tau\tilde{\phi}_k\, e^{i(k-q)x}\left(e^{k\,l_s^2\,\partial_\tau}a^2\right) \\
&\quad + \partial_\tau\tilde{\phi}_q\,\partial_\tau\tilde{\phi}_k\, e^{i(k+q)x}\left(e^{k\,l_s^2\,\partial_\tau}a^2\right) + \partial_\tau\tilde{\phi}_q^\dagger\,\partial_\tau\tilde{\phi}_k^\dagger\, e^{-i(k+q)x}\left(e^{-k\,l_s^2\,\partial_\tau}a^2\right) \\
&\quad + \partial_\tau\tilde{\phi}_q\,\partial_\tau\tilde{\phi}_k^\dagger\, e^{-i(k-q)x}\left(e^{-k\,l_s^2\,\partial_\tau}a^2\right)\Bigg].
\end{aligned}
$$

Using the Dirac delta function

$$\int dx\, e^{i(k\pm q)x} = 2\pi\delta(k \pm q), \tag{C.27}$$

and the a property of the real field: $\tilde{\phi}_k^\dagger = \tilde{\phi}_{-k}$, the action becomes

$$\tilde{S}_{time} \;=\; V\int d\tau dk \frac{1}{4}\left[\partial_\tau\tilde{\phi}_{-k}\,\partial_\tau\tilde{\phi}_k\, e^{kl_s^2\partial_\tau}a^2 \;+\; \partial_\tau\tilde{\phi}_k\,\partial_\tau\tilde{\phi}_{-k}\, e^{-kl_s^2\partial_\tau}a^2\right].$$

Expanding the exponential term, one obtains

$$\begin{aligned} e^{\pm kl_s^2\partial_\tau}a^2(\tau) &= \left[1 \pm kl_s^2\partial_\tau + \frac{1}{2}\left(kl_s^2\partial_\tau\right)^2 \pm \frac{1}{3!}\left(kl_s^2\partial_\tau\right)^3 + ...\right]a^2(\tau). \\ &\simeq a^2(\tau \pm kl_s^2). \end{aligned} \tag{C.28}$$

Therefore

$$\tilde{S}_{time} \;=\; V\int_{|k|<k_0} d\tau dk \frac{1}{2}\partial_\tau\tilde{\phi}_{-k}\,\partial_\tau\tilde{\phi}_k\, \left[\frac{a^2(\tau + kl_s^2) + a^2(\tau - kl_s^2)}{2}\right].$$

VITAE

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Presentations

1. Noncommutative inflation: The Third Thai School & ThaiPhysUniverse Symposium in Thailand, Konkhan University, Konkhan (20 October 2004).
2. Double inflation: XI Vietnam School of Physics, Danang, Vietnam (29 December 2004).
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International Schools

1. Short Course on Cosmology, Chulalongkorn University, Bangkok, 16 - 27 January 2006.
2. XI Vietnam School of Physics, Danang, Vietnam 27 December 2004 - 7 January 2005.
3. Short Course on Selforganization in Complex Systems, Chulalongkorn University, Bangkok, 13 - 24 September 2004.
4. Short Course on Conformal Field Theory, Chulalongkorn University, Bangkok, 13 - 22 September 2004.
5. The Third Thai School & ThaiPhysUniverse Symposium in Thailand, Konkhan University, Konkhan, 10 - 20 October 2004.
6. Short Course on Non-commutative Inflation, Chulalongkorn University, Bangkok, 19 - 30 July 2004.

7. II Tah Poe School on Cosmology, Naresuan University, Phitsanulok, 17 - 25 April 2003.

8. IX Vietnam School of Physics, Hue, Vietnam, 30 December 2002 - 10 January 2003.

9. I Tah Poe School on Cosmology, Naresuan University, Phitsanulok, 1 - 8 April 2002.